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Restrictions of Ultrafilters and Ultrapower Embeddings from Generic Extensions to their Ground Model

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Contents

0.1 Introduction

Let P be a forcing notion, and assume that $G \subseteq P$ is generic over V. Assume that a cardinal κ is measurable in $V[G]$. Consider the following questions:

1. What are the normal measures carried by κ in $V[G]$? Does every such measure extend a normal measure on κ in V?

2. Given such a normal measure on κ, $W \in V[G]$, let–

$$j_W : V[G] \to \text{Ult}(V[G], W) \simeq M[j_W(G)]$$

be its ultrapower embedding[1]. Is $j_W \restriction_V$ an iteration of V (by its measures or extenders)? Is $j_W \restriction_V$ a definable class of V (possibly with parameters)?

The first question is part of a larger body of work regarding the characterization and possible structure of the normal measures, carried by a cardinal κ, in forcing extensions which preserve its measurability. A very partial list of landmark results in this area include the works of Kunen-Paris [17], where the maximal possible number, κ^{++}, of normal measures on κ, is obtained; Friedman-Magidor [6], where it is proved that any intermediate value $1 < \lambda < \kappa^{++}$ can be obtained as the number of normal measures on κ; and Ben-Neria, [4], [3], where it is shown that every well-founded order can be realized as the Mitchell order on κ.

The second question is motivated by key results and ideas from inner model theory. Assume that the core mode $\mathcal{K}$ exists and $j : V \to N$ is an arbitrary elementary embedding, where N is transitive. Under limitations on the variety of large cardinals available in the universe, the restriction $j \restriction_{\mathcal{K}}$ is an iteration of $\mathcal{K}$ by its measures and extenders. For instance, if there is no inner model with a cardinal α of Mitchell order $o(\alpha) = \alpha^{++}$, then, by results of Mitchell [20], $j \restriction_{\mathcal{K}}$ is an iteration of $\mathcal{K}$ by its measures; assuming that there is no inner model with a strong cardinal, $j \restriction_{\mathcal{K}}$ is an iteration of $\mathcal{K}$ by its extenders [16]. In our context, assuming that $V = \mathcal{K}$ is the core model and $G \subseteq P$ is generic over it, the ultrapower embedding $j_W : V[G] \to M[H]$ restricts to an iteration of $V = \mathcal{K}$, provided that there is no inner model with a Woodin cardinal (see [21]).

The inner model theoretic approach has two major shortcomings. First, it limits the strength of the large cardinal properties available in the universe. Second, it imposes severe limitations on

[1] By elementarity, $\text{Ult}(V[G], W)$ is isomorphic to a model of the form $M[j_W(G)]$, where $j_W(G)$ is generic for $j_W(P)$ over some ground model M.

the ground model (namely, assuming that it is the core model). This raises the question, to what extent the inner model theoretic assumptions can be weakened while providing similar results.

An example for an approach which demonstrates this was given by Hamkins in [13]. Assuming that the forcing P has a gap below κ (see theorem 1.0.2 and the remarks preceding it), every normal measure $W \in V[G]$ is amenable to V, namely extends a normal measure $W \cap V \in V$ in V; furthermore, the embedding $j_W \restriction_V$ is definable in V. However, this method does not apply to substantial class of forcing notions, iterations of Prikry type forcings; such forcings might not admit a gap.

The iteration scheme for Prikry type forcings was first introduced by M. Magidor in his celebrated paper [18]. Its earliest application was to produce a model where the least strongly compact cardinal is the least measurable cardinal, settling a question of Tarski.

In this work, we would like to explore new tools for dealing with questions $1, 2$ above, while minimizing - usually completely omitting - any inner model theoretic assumptions. We concentrate mainly on iterations of Prikry forcings, for which the current tools do not necessarily apply. We believe that the same tools could be implemented in other forcing arguments, and provide a viable solution whenever large cardinals beyond the scope of inner model theory are used.

Similarly to any forcing construction which involves iterated forcing, iterations of Prikry type forcings can be done with respect to various supports. We concentrate on the main three supports in this context, which are the Full-support iteration (Magidor iteration), Easton support iteration (Gitik iteration) and Nonstationary support iteration. It turns out that the situation changes drastically between different supports.

In chapter 1 we present some tools for dealing with question about amenability of W to V and definability of $j_W \restriction_V$ in V, laying the foundations for the next chapters, where the focus will be on iterations of Prikry forcings.

In chapter 2, we concentrate on the nonstationary support iteration of Prikry forcings below a measurable limit of measurables κ. Assuming $\mathrm{GCH}_{\leq\kappa}$, we prove that the normal measures carried by κ in the generic extension are in bijection with normal measures of Mitchell order 0 on κ in V. We prove that the restriction of any ultrapower embedding from $V[G]$ to the ground model, is an iteration of V by normal measures only, and provide a sufficient condition for its definability in V. As an application, we prove that it is consistent that a strongly compact cardinal carries a unique normal measure, starting from a model where a supercompact exists, GCH holds and the Mitchell order is linear (the assumptions that GCH holds and the Mitchell order is linear can be weakened, see [8]). The results in this chapter are based on joint papers with Gitik, [8] and [9].

In chapter 3, we deal with the Full support iteration of Prikry forcings (Magidor iteration) below κ. As above, we characterize all the normal measures on κ in $V[G]$, proving that the set of all such measures is in bijection with the set of normal measures on κ in V. This result was first observed by Ben-Neria in [2]; we replace the inner model theoretic assumptions in the argument with the assumption that $\mathrm{GCH}_{\leq\kappa}$ holds in V. We then prove that every normal measure on κ in $V[G]$ restricts to an iterated ultrapower of V by its normal measures, and study its definability. The results in this chapter are base on [15].

In chapter 4, we study the Easton support iteration of Prikry-type forcings. It turns out that the situation is radically different from the Nonstationary and Full support iterations. We prove that, regardless of the number of normal measures in V, the number of normal measures on κ in $V[G]$ is the maximal possible number, $(2^{\kappa})^{+}$. We show that the restriction of an ultrapower embedding from $V[G]$ to V is not necessarily an iteration by normal measures only, as it might involve extenders. Nevertheless, we isolate a class of normal measures on κ in $V[G]$ ("simply generated measures") which behave similarly to the nonstationary support iteration. The results in this chapter are based on a joint work with Gitik, [10].

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Chapter 1

Background

Let P be a forcing notion over the ground model V. Assume that $G \subseteq P$ is generic over V, κ is a cardinal which is measurable in $V[G]$, and $W \in V[G]$ is a measure (not necessarily normal) on κ in $V[G]$.

Consider the ultrapower embedding $j_W : V[G] \to \mathrm{Ult}(V[G], W)$. By elementarity, $\mathrm{Ult}(V[G], W)$ has a transitive collapse of the form $M[H]$, where $H = j_W(G)$ is $j_W(P)$-generic over some ground model M which has the form–

$$M = \bigcup_{\alpha \in \mathrm{On}} j_W(V_\alpha)$$

M is not necessarily a definable class of V; furthermore, it doesn't have to be the case that $M \subseteq V$. The embedding $j_W \restriction_V : V \to M$ is clearly elementary. Similarly to the above,

$$j_W \restriction_V = \bigcup_{\alpha \in \mathrm{On}} j_W \restriction_{V_\alpha}$$

$j_W \restriction_V$ is not necessarily a class of V; $W = U \cap V$ is not necessarily a measure on κ in V - possibly, κ is not even measurable in V. Even in the case this holds, and $U = W \cap V$ belongs to V, it is not necessarily the case that j_W extends the ultrapower embedding of U, $j_U : V \to M$.

Let us sketch some of the main tools for dealing with questions 1, 2 presented in the introduction.

We begin with the following theorem, which deals with the case where $M \subseteq V$ holds. The statement of the theorem below was brought to the author's attention by Schlutzenberg. The proof presented below, is mainly due to Goldberg. An independent argument is due to Schlutzenberg (see lemma 2.2 in [22]).

Theorem 1.0.1. *Assume that P is a set forcing and $M \subseteq V$. Let $j \colon V[G] \to M[H]$ be an elementary embedding definable in $V[G]$ (possibly from parameters). Then M and $j\restriction_V$ are definable classes of V (with parameters).*

Proof. We first argue that M is a definable class of V (with parameters). Indeed, by the remarks above,

$$M = \bigcup_{\alpha \in \mathrm{On}} j(V_\alpha)$$

Since $M \subseteq V$, it follows that M is an increasing union of sets in V. For each ordinal α, let $p_\alpha \in P$ be a condition which decides the value of $j(V_\alpha)$. Since P is a set forcing, there exists $p^* \in P$ such that, for unboundedly many ordinals α, p^* decides $j(V_\alpha)$. Thus p^* forces that M is definable as the union, along a class of ordinals (ordinals $\alpha \in \mathrm{ON}$ for which $p_\alpha = p^*$), of the decided values of $j(V_\alpha)$. Such p^* can be found above any condition $p \in P$, so we can assume that $p^* \in G$ and thus M is definable in V from p^*.

The second step is the proof that $j\restriction_{\mathrm{ON}}$ is definable in V. In [12] it is proved that any pair of elementary embeddings, $i_0, i_1 \colon V \to M$ agree on the ordinals (definability in V is not required). In particular, $j\restriction_{\mathrm{ON}}$ is the unique restriction to the ordinals of an elementary embedding which satisfies:

1. It is definable in any generic extension of V (with a generic set for P), by the same formula which defines j in $V[G]$.

2. Its target when restricted to V is M.

Therefore, $j\restriction_{\mathrm{ON}}$ is definable in V. Let us argue now that j is amenable to V, in the sense that, for every set $X \in V$, $j\restriction_X \in V$. Fix a set $X \in V$. let $f \colon \mu \to X$ be a bijection, for some cardinal μ. Then $j\restriction_X = j(f) \circ j\restriction_\mu \circ f^{-1}$ and thus belongs to V.

Amenability of j to V concludes the proof, since, arguing as above, there exists $p^* \in G$ which decides $j\restriction_{V_\alpha}$ for a definable proper class of $\alpha \in \mathrm{ON}$. So $j\restriction_V$ is definable from parameters in V.

□

Corollary 1.0.1. *Assume that $W \in V[G] \to M[H]$ is a normal measure on a measurable cardinal κ. Assume that $M \subseteq V$. Then W is amenable to V.*

Proof. By the previous theorem, $j_W \restriction_V$ is definable in V. Then–

$$W \cap V = \{X \subseteq \kappa \colon \kappa \in j_W \restriction_V (X)\}$$

and this definition is carried out in V.

□

The condition $M \subseteq V$ above does not necessarily hold in general. By Hamkins [13], assuming some structure on the forcing notion P ensures this condition. We say that a forcing notion P has a gap at a cardinal δ if P can be factored to the form $P_1 * \underset{\sim}{P}_2$, where P_1 is nontrivial and of cardinality strictly below δ, and–

$$\Vdash_{P_1} \text{"}\underset{\sim}{P}_2 \text{ is } (\delta+1)\text{-strategically closed."}$$

Theorem 1.0.2. *(Gap Forcing Theorem, Hamkins, [13]) Let $\delta < \kappa$ be a cardinal, and assume that P has a gap at δ. Let $j \colon V[G] \to M[j(G)]$ be an arbitrary elementary embedding such that–*

1. *$crit(j) = \kappa$.*
2. *$V[G] \vDash {}^{\delta}M[j(G)] \subseteq M[j(G)]$.*
3. *$M[j(G)] \subseteq V[G]$.*

Then:

1. *$M \subseteq V$. Moreover, $M = V \cap M[j(G)]$.*
2. *If j is amenable to $V[G]$ (namely, for every $X \in V[G]$, $j \restriction_X \in V[G]$) then $j \restriction_V$ is amenable to V (namely, for every $X \in V$, $j \restriction_X \in V$).*
3. *If j is definable (possibly from parameters) in $V[G]$, then $j \restriction_V$ is definable (from the names of those parameters) in V.*

Theorems 1.0.1 and 1.0.2 provide sufficient conditions for a vast class of forcing notions. However, they cannot be applied to iterations of Prikry-type forcings. In general, such iterations below a cardinal κ do not admit a gap below it. Furthermore, given $W \in V[G]$ and its corresponding ultrapower embedding $j_W \colon V[G] \to M[j_W(G)]$, it is not necessarily true that $M \subseteq V$. Different tools are required for dealing with such forcings, and this is the main subject of this work.

We begin with a criterion for amenability of a measure $W \in V[G]$ to V. The main idea behind it is an analysis of the fresh[1] subsets added in the extension from V to $V[G]$. Recall that a set of ordinals $A \in V[G]$ with supremum α is called fresh over V, if $A \notin V$, but for every $\beta < \alpha$, $A \cap \beta \in V$.

Proposition 1.0.2. *(Gitik, K. [8]) Let $V[G]$ is a generic extension of V by a forcing P. Suppose that κ is a measurable cardinal in $V[G]$ and W is a $\kappa-$complete ultrafilter over κ. Let $U = V \cap W$. Then $U \in V$ if the following hold:*

1. *all cardinals of V in the interval $[\kappa, (2^\kappa)^V]$ are preserved,*
2. *no fresh subsets are added to a cardinal λ, $\kappa \leq \lambda \leq (2^\kappa)^V$.*

Proof. Let $\delta = (2^\kappa)^V$. For every bijection $f : \delta \leftrightarrow \mathcal{P}(\kappa)$ in V, set–

$$X_f = \{\alpha < \delta \mid f(\alpha) \in W\}.$$

Clearly, if for some such f, $X_f \in V$, then also $U \in V$. Suppose that for every bijection $f : \delta \to \mathcal{P}(\kappa)$, $X_f \notin V$. For every such f, let $\alpha_f \leq \delta$ be the least α such that $X_f \cap \alpha \notin V$. Set–

$$\alpha^* = \min\{\alpha_f \mid f : \delta \leftrightarrow \mathcal{P}^V(\kappa), f \in V\}.$$

Claim. $\alpha^* < \kappa$.

Proof. Suppose otherwise. By the first two assumption of the theorem, α^* cannot be a cardinal in the interval $[\kappa, \delta]$. So, there is a cardinal $\eta, \kappa \leq \eta < \delta$ such that $\eta < \alpha^* < \eta^+$. Pick, in V, $g : \delta \leftrightarrow \delta$ such that $g \restriction \eta$ maps η onto α^*. Then $X_{f \circ g} \cap \eta \notin V$, and hence, $\alpha_{f \circ g} \leq \eta$. But $\eta < \alpha^* \leq \alpha_{f \circ g}$. Contradiction. □

So, $\alpha^* < \kappa$. Fix $f : \delta \leftrightarrow \mathcal{P}(\kappa)$ with $\alpha_f = \alpha^*$. So,

$$X_f \cap \alpha^* = \{\alpha < \alpha^* \mid f(\alpha) \in W\} \notin V$$

Work in $V[G]$. Set $A_0 = \bigcap_{\alpha \in X_f \cap \alpha^*} f(\alpha)$ and $A_1 = \bigcap_{\alpha \in \alpha^* \setminus X_f} \kappa \setminus f(\alpha)$. Both sets are in W due to $\kappa-$completeness. So, $A = A_0 \cap A_1 \in W$, as well. Pick $\zeta \in A_0 \cap A_1$. Then–

$$\{\alpha < \alpha^* \mid \zeta \in f(\alpha)\} = X_f \cap \alpha^*,$$

but clearly, $\{\alpha < \alpha^* \mid \zeta \in f(\alpha)\}$ is in V. Thus, $X_f \cap \alpha^* \in V$. Contradiction. □

[1]The notion of a fresh set of ordinals in due to Hamkins.

Given $W \in V[G]$, amenability to V, namely having $U = W \cap V \in V$, provides a one-step factorization of $j_W \restriction_V$. Denote by M_U the transitive collapse of $\mathrm{Ult}(V[G], U)$. There exists a natural elementary embedding $k\colon M_U \to M$,

$$k([f]_U) = [f]_W$$

for every $f \in V$ with $\mathrm{dom}(f) = \kappa$.

Claim 1.0.3. *The embedding $k\colon M_U \to M$ defined above is elementary, and $j_W \restriction_V = k \circ j_U$.*

Proof. k is well defined, since, if $[f]_U = [g]_U$, then $\{x < \kappa\colon f(x) = g(x)\} \in U$, and thus this set belongs to W. So $[f]_W = [g]_W$. Similarly, k respects $\in$. Finally, assume that $\varphi(a_1, \ldots, a_n)$ is a formula and $f_1, \ldots, f_n$ are functions. Then:

$$\begin{aligned} M_U \vDash \varphi([f_1]_U, \ldots [f_n]_U) &\iff \{x < \kappa\colon V \vDash \varphi(f_1(x), \ldots, f_n(x)\} \in U \\ &\iff \{x < \kappa\colon V \vDash \varphi(f_1(x), \ldots, f_n(x)\} \in W \\ &\iff M[H] \vDash M \vDash \varphi([f_1]_W, \ldots [f_n]_W) \\ &\iff M \vDash \varphi([f_1]_W, \ldots [f_n]_W) \end{aligned}$$

Finally, we argue that $j_W \restriction_V = k \circ j_U$. Fix $x \in V$, and let $c_x\colon \kappa \to V$ be the function with constant value x. Then–

$$k(j_U(x)) = k([c_x]_U) = [c_x]_W = j_W(x)$$

□

Remark 1.0.4. *Assume that $U = W \cap V$ is a normal measure in V (This will typically be the case in this work). Then the embedding $k\colon M_U \to M$ above has critical point κ if and only if W is non-normal. Indeed, $k(\kappa) = k([Id]_U) = [Id]_W$, which is strictly above κ if and only if W is non-normal.*

In the upcoming chapters, we consider iterations of Prikry forcings. We utilize proposition 1.0.2 and some basic properties of the embedding k, in order to characterize all the normal measures on κ in the generic extension. We then factor k into an iterated ultrapower, proving that for every normal measure $W \in V[G]$, its restriction to V is an iterated ultrapower.

It turns our that the support used in the iteration has a substantial role in the characterization of normal measures and the structure of their ultrapowers restricted to V. We consider the three main supports - the Nonstationary support, Full support and Easton support, devoting one chapter to each one of them.

Chapter 2

The Nonstationary Support

2.1 Introduction

We consider in this chapter the Nonstationary-support iteration of Prikry forcings. The use of the Nonstationary-support in iterated forcings was introduced by Jensen ([1]), and was later utilized in the celebrated work of Friedman and Magidor [6] for controlling the number of normal measures on a measurable cardinal κ.

Our first goal in this chapter is to analyze all the normal measures carries by a measurable limit of measurables κ, after forcing with the nonstationary support iteration of Prikry forcings below it. We denote this forcing by P and assume that $G \subseteq P$ is generic for it over V. For every normal measure U on κ in V of Mitchell order 0, we will define a normal measure $U^* \in V[G]$ which extends it (using the standard methods of [7]), and prove–

Theorem 2.1.1. *Assume $GCH_{\leq\kappa}$. Every normal measure $W \in V[G]$ on κ has the form U^* for some normal measure $U \in V$ of Mitchell order 0. Furthermore, U^* is the unique normal measure in $V[G]$ which extends U.*

We then proceed and prove that, for every normal measure on κ, $W \in V[G]$, the restriction $j_W \restriction_V$ is an iterated ultrapower of V.

Theorem 2.1.2. *Assume $GCH_{\leq\kappa}$. Then for every normal measure $W \in V[G]$ on κ, $j_W \restriction_V$ is an iterated ultrapower of V by normal measures.*

We also describe the iteration and provide a sufficient condition for its definability as a class of V. For instance, we prove the following:

Corollary 2.1.3. *Assume that the Mitchell order is linear (or even, uniqueness of normal measures of Mitchell order* 0*) and that* $GCH_{\leq\kappa}$ *holds. Then there exists a unique normal measure* $W \in V[G]$ *on* κ*, and* $j_W \restriction_V$ *is a definable class of* V*.*

This chapter is structured as follows: In section 1, we present the forcing and its basic properties. We rely on the work of Ben-Neria and Unger in [5], where a framework for the nonstationary support iteration of Prikry type forcing was developed. In section 2, we characterize all the normal measures $W \in V[G]$ on κ, using and extending results from [5] and [8]; more specifically, we prove that every such measure is the unique extension of some normal measure of Mitchell order 0 on κ in V. In section 3, we present the structure of $j_W \restriction_V$ as an iterated ultrapower, and provide a sufficient condition for its definability in V. In section 4, we study iterated ultrapowers of V in general, developing tools for computation of cofinalities, in V, of ordinals which become inaccessibles at some stage in an iteration; we apply those tools to simplify the presentation of $j_W \restriction_V$ as an iteration of V. Finally, in section 5, we apply the above tools and construct a model where the least strongly compact κ is the least measurable and carries a unique normal measure, starting from a model where κ is supercompact, $\mathrm{GCH}_{\leq\kappa}$ holds and the Mitchell order is linear.

We assume throughout this chapter that $\mathrm{GCH}_{\leq\kappa}$ holds in V.

2.2 The Forcing

Definition 2.2.1. *An iteration* $\langle P_\alpha, \underset{\sim}{Q}_\beta \colon \alpha \leq \kappa\ ,\ \beta < \kappa \rangle$ *is called a nonstationary support iteration of Prikry-type forcings if and only if, for every* $\alpha \leq \kappa$ *and* $p \in P_\alpha$*,*

1. *p is a function with domain* α *such that for every* $\beta < \alpha$*,* $p \restriction_\beta \in P_\beta$*, and* $p\restriction_\beta \Vdash p(\beta) \in \underset{\sim}{Q}_\beta$ *and* $\langle \underset{\sim}{Q}_\beta, \underset{\sim}{\leq}_{Q_\beta}, \underset{\sim}{\leq}^*_{Q_\beta} \rangle$ *is a Prikry-type forcing.*
2. *If* $\alpha \leq \kappa$ *is inaccessible, then* $supp(p) \cap \alpha$ *is nonstationary in* α *(where* $supp(p) \subseteq \alpha$ *is the complement of the set* $\{\beta < \alpha \colon p_{\restriction_\beta} \Vdash p(\beta) \text{ is trivial}\}$*). In other words, there exists a club* $C \subseteq \alpha$ *such that for every* $\beta \in C$*,* $p\restriction_\beta \Vdash p(\beta)$ *is trivial.*

Suppose that $p, q \in P_\alpha$*. Then* $p \geq q$*, which means that* p *extends* q*, holds if and only if:*

1. $supp(q) \subseteq supp(p)$*.*
2. *For every* $\beta \in supp(q)$*,* $p \restriction_\beta \Vdash p(\beta) \geq_\beta q(\beta)$ *(where* $\geq_\beta$ *is the order of* Q_β*).*

3. *There is a finite subset $b \subseteq supp(q)$, such that for every $\beta \in supp(q) \setminus b$, $p \restriction_\beta \Vdash p(\beta) \geq^*_\beta q(\beta)$ (where $\geq^*_\beta$ is the direct extension order of Q_β).*

If $b = \emptyset$, we say that p is a direct extension of q, and denote it by $p \geq^* q$.

We consider a nonstationary support iteration of Prikry forcings, $\langle P_\alpha, \underset{\sim}{Q}_\beta \colon \alpha \leq \kappa \ , \ \beta < \kappa \rangle$. Throughout this chapter, we denote the forcing P_κ by P. Let $\Delta \subseteq \kappa$ the set of measurable cardinals below κ in V. Assume that $\alpha \in \Delta$ and P_α has been defined. Assume that $\underset{\sim}{U}^*_\alpha$ is a P_α-name for a normal measure on α in V^{P_α} (we will prove that at least one such measure exists). Let $\underset{\sim}{Q}_\alpha$ be the Prikry forcing with $\underset{\sim}{U}^*_\alpha$. If α is not measurable in V, $\underset{\sim}{Q}_\alpha$ is the trivial forcing.

We did not specify the normal measure U^*_α which is used at stage α. As we will prove, each such measure in V^{P_α} is the unique extension of a normal measure U_α of Mitchell order 0 in V. Let $\underset{\sim}{\mathcal{U}} = \langle \underset{\sim}{U}_\alpha \colon \alpha \in \Delta \rangle$ be a sequence of names, such that, for every $\alpha \in \Delta$, $\underset{\sim}{U}_\alpha$ is forced by the weakest condition in P_α to be $\underset{\sim}{U}^*_\alpha \cap V$. Given $G \subseteq P_\kappa$ generic over V, let $\mathcal{U} = \langle U_\alpha \colon \alpha \in \Delta \rangle$ be the interpretation of the names in $\underset{\sim}{\mathcal{U}}$. Then $\mathcal{U}$ is a sequence of measures in V, but $\mathcal{U}$ itself does not necessarily belong to V. Since $\mathcal{U}$ depends on G, a more accurate notation would be $\mathcal{U}_G$, but most of the time G will be clear from the context.

We adopt the following notation: For every $p \in P_\kappa$ and $\alpha \in \Delta$, let $\underset{\sim}{t}^p_\alpha, \underset{\sim}{A}^p_\alpha$ be P_α-names such that $p \restriction_\alpha \Vdash p(\alpha) = \langle \underset{\sim}{t}^p_\alpha, \underset{\sim}{A}^p_\alpha \rangle$.

An iteration of Prikry-type forcings with nonstationary support was studied in [5]. The following key property was proved:

Lemma 2.2.2. *$P = P_\kappa$ satisfies the Prikry property.*

The proof relies on a fusion property which holds in our iteration. We will use the formulation of this property as it is stated and proved in [5]:

Lemma 2.2.3. *(Fusion Lemma) Let $\lambda \leq \kappa$ be a limit ordinal, and assume that $p \in P_\lambda$. Suppose that $e \colon \lambda \to V$ is a function such that for every $\alpha < \lambda$, $e(\alpha)$ is a $P_{\alpha+1}$-name, such that,*

$$p \restriction_{\alpha+1} \Vdash \text{"}e(\alpha) \text{ is a dense open subset of } P_\lambda \setminus (\alpha+1) \text{ above } p \setminus (\alpha+1)\,,$$
$$\text{with respect to the direct extension order."}$$

Assume also that $\nu < \lambda$ is an ordinal. Then there exist $p^* \geq^* p$ which satisfies $p^* \restriction_\nu = p \restriction_\nu$, and a club $C \subseteq \lambda$, such that for every $\alpha \in C$,

$$p^* \restriction_{\alpha+1} \Vdash p^* \setminus (\alpha+1) \in e(\alpha)$$

The Fusion Lemma will be applied repeatedly in this paper, and is standard in nonstationary support iterations. For sake of completeness, we provide the proof.

Proof. As in [5], we focus first on the case where λ is an inaccessible cardinal. The other case is simpler since an inverse limit is taken at λ.

We construct a sequence $\langle p_\xi \colon \xi < \lambda\rangle$ of conditions in P_λ, a sequence $\langle \nu_\xi \colon \xi < \lambda\rangle$ of ordinals below λ and a sequence of clubs $\langle C_\xi \colon \xi < \lambda\rangle$, such that,

1. The sequence $\langle p_\xi \colon \xi < \lambda\rangle$ is increasing with respect to direct extensions.
2. The sequence $\langle \nu_\xi \colon \xi < \lambda\rangle$ is increasing, continuous and unbounded in λ.
3. For every $\xi < \lambda$, $C_\xi \cap \operatorname{supp}(p_\xi) = \emptyset$.
4. For every $\xi < \lambda$, $\{\nu_\eta \colon \eta < \lambda\}$ is disjoint from the support of p_ξ.
5. For every $\xi < \lambda$, $p_\xi \restriction_{(\nu_\xi+1)} \Vdash p_\xi \setminus (\nu_\xi + 1) \in e(\nu_\xi)$.
6. Whenever $\eta < \xi < \lambda$,
   (a) $\mu_\xi \in C_\eta$.
   (b) $p_\xi \restriction_{\nu_\eta+1} = p_\eta \restriction_{\nu_\eta+1}$.
   (c) $p_\xi \restriction_{\nu_\eta+1} \Vdash p_\xi \setminus (\nu_\eta + 1) \geq^* p_\eta \setminus (\nu_\eta + 1)$.

Take $p_0 = p$, C_0 a club disjoint from $\operatorname{supp}(p_0)$, and $\nu_0 > \nu$ in C_0.

Successor stages: Suppose that the construction is done up, and including, some $\xi < \lambda$, and let us construct $p_{\xi+1}$ and $\nu_{\xi+1}$. Define–

$$\nu_{\xi+1} = \min\left(\bigcap_{\eta<\xi+1} C_\eta \setminus (\nu_\xi + 1)\right)$$

Let us construct $p_{\xi+1}$. First, we require $p_{\xi+1} \restriction_{\nu_{\xi+1}+1} = p_\xi \restriction_{\nu_{\xi+1}+1}$. Now, there exists a $P_{\nu_{\xi+1}}$-name for a direct extension of $p_\xi \setminus \nu_{\xi+1}$ which is forced, by $p_\xi \restriction_{\nu_{\xi+1}+1}$, to belong to $e(\nu_{\xi+1})$. Let σ be this name, and take $p_{\xi+1} \setminus \nu_{\xi+1} = \sigma$. There exists a $P_{\nu_{\xi+1}}$-name $\underset{\sim}{C}$ for a club in λ disjoint from $\operatorname{supp}(\sigma)$; Since λ is inaccessible, $P_{\nu_{\xi+1}}$ is λ-c.c., so there exists a club in λ, $C' \in V$, which is forced to be a subset of $\underset{\sim}{C}$. Hence $p_{\xi+1}$ has a club $C_{\xi+1} \in V$ disjoint from its support, and is a legitimate condition in P_λ.

Limit stages: Suppose that $\xi < \lambda$ is a limit ordinal. Set $\nu_\xi = \cup_{\eta<\xi}\nu_\eta$. For every $\eta < \xi$, ν_ξ is a limit point of C_η, and thus $\nu_\xi \notin \text{supp}\,(p_\eta)$. Let us construct p_ξ. We construct it such that $\nu_\xi \notin \text{supp}(p_\xi)$. First, we set–

$$p_\xi \restriction_{\nu_\xi} = \bigcup_{\eta<\xi} p_\eta \restriction_{\nu_\eta+1}$$

note that $\langle \nu_\eta \colon \eta < \xi \rangle$ is disjoint from the support of $p_\xi \restriction_{\nu_\xi}$, so $p_\xi \restriction_{\nu_\xi} \in P_{\nu_\xi}$ holds even if ν_ξ is inaccessible. Also, $p_\xi \restriction_{\nu_\xi+1}$ forces that $\langle p_\eta \setminus (\nu_\xi+1) : \eta < \xi \rangle$ is an increasing sequence with respect to direct extension in $P \setminus (\nu_\xi + 1)$, which is forced to be $|\nu_\xi|^+$-closed (so it's definitely more than ξ-closed). Thus, there exists an upper bound. Take $p_\xi \setminus (\nu_\xi + 1)$ to be a name, which is forced, by $p_\xi \restriction_{\nu_\xi+1}$, to be a direct extension of the upper bound which belongs to $e(\nu_\xi)$. Pick $C_\xi \subseteq \lambda$ as a club disjoint from $\text{supp}\,(p_\xi)$.

This finishes the construction. Finally, set–

$$p^* = \bigcup_{\xi<\lambda} p_\xi \restriction_{\nu_\xi+1}$$

Let $C = \{\nu_\xi \colon \xi < \lambda\} \subseteq \triangle_{\xi<\lambda} C_\xi$. Then, by our construction, $C \subseteq \lambda$ is a club disjoint from $\text{supp}\,(p^*)$. Therefore, p^* is a legitimate condition in P_λ. Also, given $\alpha \in C$, let $\xi < \lambda$ be such that $\alpha = \nu_\xi$. Then $p^* \restriction_{\alpha+1} = p_\xi \restriction_{\alpha+1}$, and thus it forces that $p^* \setminus (\alpha+1) \geq^* p_\xi \setminus (\alpha+1) \in e(\alpha)$, as desired.

Now, let us adjust the proof to the case where λ is not inaccessible. Fix in advance an increasing, continuous and cofinal sequence $\langle \nu_\xi \colon \xi < \text{cf}(\lambda) \rangle$ in λ, such that $\nu_0 > \text{cf}(\lambda)$. Now construct a $\leq^*$-increasing sequence of conditions $\langle p_\xi \colon \xi < \text{cf}(\lambda) \rangle$. In successor steps, assuming that p_ξ has been constructed, pick $p_{\xi+1}$ such that–

$$p_{\xi+1} \restriction_{\nu_{\xi+1}+1} = p_\xi \restriction_{(\nu_{\xi+1}+1)}$$

and $p_{\xi+1} \restriction_{\nu_{\xi+1}+1} \Vdash p_{\xi+1} \setminus (\nu_{\xi+1}+1) \in e\,(\nu_{\xi+1})$. In limit steps, say for limit $\xi < \text{cf}(\lambda)$, choose p_ξ such that–

$$p_\xi \restriction_{\nu_\xi} = \bigcup_{\eta<\xi} p_\eta \restriction_{\nu_\eta+1}$$

and $p_\xi \restriction_{\nu_\xi}$ forces that $p_\xi \setminus \nu_\xi$ is a $\leq^*$-upper bound of $\langle p_\eta \setminus \nu_\xi \colon \eta < \xi \rangle$ (this is the main difference from the case where λ is regular. Note that the direct extension order of $P_\lambda \setminus \nu_\xi$ is more than ξ-closed, since $\nu_\xi > \xi$). Then, direct extend further above $\nu_\xi + 1$ such that $p_\xi \restriction_{\nu_\xi+1} \Vdash p_\xi \setminus \nu_\xi + 1 \in e(\nu_\xi)$.

Finally, set $p^* = \bigcup_{\xi<\text{cf}(\lambda)} p_\xi \restriction_{\nu_\xi+1}$. $\square$

The following claim takes care of dense open subsets of P_κ (not necessarily with respect to direct extensions).

Claim 2.2.4. *Let $\lambda \le \kappa$ be a limit ordinal and let $D \subseteq P_\lambda$ be a dense open subset of P_λ. Assume that $p \in P_\lambda$ and $\nu < \lambda$. Then there exist $p^* \ge^* p$ and a club $C \subseteq \lambda$, such that $p^* \restriction_\nu = p \restriction_\nu$, and, for every $p^* \le q \in D$,*

$$q \restriction_{\gamma+1} {}^\frown p^* \setminus (\gamma+1) \in D$$

where $\gamma \in C$ is the first coordinate for which–

$$q \restriction_{\gamma+1} \Vdash \text{"}q \setminus \gamma \text{ is a direct extension of } p^* \setminus \gamma\text{"}$$

Proof. Fix a non-measurable $\xi < \lambda$ and $G_\xi \subseteq P_\xi$ generic over V such that $p \restriction_\xi \in G_\xi$. Given $p \restriction_\xi \le q \in G_\xi$, we define a subset of $P_\lambda \setminus \xi$ which is $\le^*$-dense open above $p \setminus \xi$:

$$e_q(\xi) = \{r \in P \setminus \xi \colon q^\frown r \in D \text{ or } (\forall r' \ge^* r,\ q^\frown r' \notin D)\}$$

Since ξ is non-measurable, the direct extension order of $P_\lambda \setminus \xi$ is more than $|G_\xi|^+$-distributive. Let $e(\xi)$ be a P_ξ-name for the set–

$$e(\xi) = \bigcap_{q \in G_\xi} e_q(\xi)$$

Then $p \restriction_\xi$ forces that $e(\xi)$ is $\le^*$-dense open above $p \setminus \xi$.

Apply lemma 2.2.3. Let $p^* \ge^* p$ be such that $p^* \restriction_\nu = p \restriction_\nu$, and there exists a club C such that, for every $\alpha \in C$,

$$p^* \restriction_{\alpha+1} \Vdash p^* \setminus (\alpha+1) \in e(\alpha)$$

Assume now that $p^* \le q \in D$. Let $\gamma \in C$ be as in the formulation of the claim. Then–

$$p^* \restriction_{\gamma+1} \Vdash p^* \setminus (\gamma+1) \in e(\gamma+1)$$

In particular,

$$q \restriction_{\gamma+1} \Vdash p^* \setminus (\gamma+1) \in e(\gamma+1)$$

Finally, since there exists a direct extension $r' = q \setminus (\gamma+1) \ge^* p^* \setminus (\gamma+1)$ such that $q \restriction_{\gamma+1} {}^\frown r' \in D$, it follows that $q \restriction_{\gamma+1} {}^\frown p^* \setminus (\gamma+1) \in D$, as desired. □

Lemma 2.2.5. *$P = P_\kappa$ preserves cardinals. It also preserves cofinalities $\ge \kappa^+$.*

Proof. P clearly preserves cardinals and cofinalities $\geq \kappa^{++}$, since it has cardinality κ^{+}.

Let us prove by induction that every cardinal $\mu \leq \kappa^{+}$ is not collapsed. For limit μ it's clear. Suppose that $\mu = \lambda^{+}$ is a successor. Split $P = P_{\lambda} * \underset{\sim}{Q}_{\lambda} * P \setminus (\lambda+1)$. The direct extension order of $P \setminus (\lambda+1)$ is more than μ-closed, so it preserves μ. Q_{λ} preserves cardinals, whether λ is measurable or not. Thus, it suffices to prove that P_{λ} preserves $\lambda^{+} = \mu$, for every $\lambda \leq \kappa$. Suppose that $\underset{\sim}{f}$ is a P_{λ}-name for an increasing function from λ to μ, and this is forced by an arbitrary condition $p \in P_{\lambda}$. We will prove that there exists an extension p^{*} of p in P_{λ} which forces that the image of $\underset{\sim}{f}$ is bounded in μ.

For every $\xi < \lambda$, define the following $P_{\xi+1}$-name for a dense open subset of $P \setminus (\xi+1)$,

$$e(\xi) = \{r \in P \setminus \xi + 1 \colon \exists \delta < \lambda^{+},\ r \Vdash \underset{\sim}{f}(\xi) < \delta\}$$

We claim that $e(\xi)$ is $\leq^{*}$-dense open. First, let us argue that this suffices. Indeed, by fusion, there exists $p^{*} \in G$ and a club $C \subseteq \lambda$ such that for every $\xi \in C$,

$$p^{*} \restriction_{\xi+1} \Vdash \exists \delta_{\xi} < \lambda^{+},\ p^{*} \setminus (\xi+1) \Vdash \underset{\sim}{f}(\xi) < \delta_{\xi}$$

and set–

$$\delta^{*} = \sup\left(\bigcup_{\xi \in C} \{\delta \colon \exists r \geq p^{*} \restriction_{\xi+1},\ r \Vdash \underset{\sim}{\delta_{\xi}} = \delta\}\right)$$

Then $\delta^{*} < \lambda^{+}$ and, since $\underset{\sim}{f}$ is increasing, $p^{*} \Vdash \operatorname{Im}(\underset{\sim}{f}) \subseteq \delta^{*} + 1$.

Let us prove that $e(\xi)$ is indeed $\leq^{*}$-dense open. Fix $\xi < \lambda$. Let $G' \subseteq P_{\xi+1}$ be generic over V, and work in $V[G']$. Denote $P' = P \setminus (\xi+1)$. Apply claim 2.2.4 for the dense open set D of conditions in P' which decide the value of $\underset{\sim}{f}(\xi)$. Given a condition $q \in P'$, there exists $q^{*} \geq^{*} q$ and a club $C \subseteq \lambda$ such that for every $q^{*} \leq p \in D$,

$$p \restriction_{\gamma+1} {}^\frown q^{*} \setminus (\gamma+1) \in D$$

where γ is the least coordinate in C above the non-direct extensions. Let–

$$\delta^{*} = \sup\left(\bigcup_{\gamma \in C} \{\delta \colon \exists s \in P'_{\gamma+1},\ s {}^\frown q^{*} \setminus (\gamma+1) \Vdash \underset{\sim}{f}(\xi) = \delta\}\right)$$

Then $q^{*} \Vdash \underset{\sim}{f}(\xi) < \delta^{*}$. $\square$

The following lemma is a minor modification of lemma 3.6 from [5].

Lemma 2.2.6. *Let $\lambda \le \kappa$ be inaccessible. Let $p \in P_\lambda$ and assume that $\underset{\sim}{f}$ is a P_λ-name for a function from λ to the ordinals. Then there exist $p^* \ge^* p$, a club $C \subseteq \lambda$ and a function $F \colon \lambda \to [Ord]^{<\lambda}$ in V, such that for every $\xi \in C$, $p^* \Vdash \underset{\sim}{f}(\xi) \in F(\xi)$.*

Proof. For each $\xi < \lambda$, consider the $P_{\xi+1}$-name for the following set–

$$e(\xi) = \{r \in P_\lambda \setminus \xi \colon \exists A \in [\mathrm{Ord}]^{<\lambda},\ r \Vdash \underset{\sim}{f}(\xi) \in A\}$$

It suffices to prove that for every $\xi < \lambda$, $e(\xi)$ is forced to be $\le^*$-dense open subset of $P_\lambda \setminus (\xi+1)$. Indeed, once we prove this, there exist $p^* \in G$ above p and a club $C \subseteq \lambda$ such that for every $\xi \in C$,

$$p^* \restriction_{\xi+1} \Vdash \exists A_\xi \in [\mathrm{Ord}]^{<\lambda},\ p^* \setminus (\xi+1) \Vdash \underset{\sim}{f}(\xi) \in A(\xi)$$

and then, for every $\xi \in C$, we can define–

$$F(\xi) = \{\gamma \colon \exists q \ge p^* \restriction_{\xi+1},\ q \Vdash \gamma \in \underset{\sim}{A}_\xi\}$$

Then $|F(\xi)| < \lambda$ for every $\xi \in C$, and $p^* \Vdash \underset{\sim}{f}(\xi) \in F(\xi)$.

Let us prove that $e(\xi)$ is $\le^*$ dense open. Fix $\xi < \kappa$. Let $G' \subseteq P_{\xi+1}$ be generic over V, and work in $V[G']$. Denote $P' = P_\lambda \setminus (\xi+1)$. It suffices to prove that given a condition $q \in P'$, there exists a direct extension $q^* \ge^* q$ and a set $A \in [\mathrm{Ord}]^{<\lambda}$ such that $q^* \Vdash \underset{\sim}{f}(\xi) \in A$.

Let $D \subseteq P'$ be the dense open set of conditions $r \in P'$ such that, for some $A \in [Ord]^{<\lambda}$, $r \Vdash \underset{\sim}{f}(\xi) \in A$. By claim 2.2.4, there exists $q^* \ge^* q$ and a club $C \subseteq D$, such that for every $q^* \le p \in D$, $p \restriction_{\gamma'+1}{}^\frown q^* \setminus (\gamma'+1) \in D$, where $\gamma' = \min(C \setminus (\gamma+1))$, and γ is the maximal coordinate in which a non-direct extension is taken in the extension $q^* \le p$.

Let us construct a direct extension $q^{**} \ge^* q^*$ with the same support as q^*. Let $\mu \in \mathrm{supp}\,(q^*)$ be a measurable, and assume that $q^{**} \restriction_\mu$ was constructed. Take an arbitrary generic $G_\mu \subseteq P'_\mu$ with $q^{**} \restriction_\mu \in G_\mu$. Denote $\mu' = \min(C \setminus (\mu+1))$. In $V[G', G_\mu]$, shrink the set $\underset{\sim}{A}^{q^*}_\mu$ to a set A such that, for each $n < \omega$, exactly one of the following holds: Either for every $s \in [A]^n$, there exists direct extension $r_s \ge^* q^* \restriction_{(\mu,\mu']}$ and a set of ordinals A_s with $|A_s| < \lambda$, such that–

$$\langle t^{q^*}_\mu{}^\frown s, A \setminus \max(s)\rangle^\frown r_s{}^\frown q^* \setminus (\mu'+1) \Vdash \underset{\sim}{f}(\xi) \in \check{A}_s$$

or, there is no such $s \in [A]^n$.

Let us prove now that q^{**} has a direct extension which belongs to $e(\xi)$. Assume otherwise. Let $p \ge q^{**}$ be a condition which decides the value of $\underset{\sim}{f}(\xi)$, and is chosen with the least number of non–direct extensions. Let $\gamma \in \operatorname{supp}(q^*)$ be the maximal coordinate in which a non-direct extension is taken, and let $\gamma' = \min\left(C \setminus (\gamma+1)\right)$. Clearly $p \geq q^*$, and by the choice of q^*,

$$p \restriction_{\gamma'+1} {}^\frown q^* \setminus (\gamma'+1) \in D$$

In particular, for some $A \in [\mathrm{Ord}]^{<\lambda}$,

$$p \restriction_{\gamma} \Vdash p \restriction_{[\gamma,\gamma']} \Vdash q^* \setminus (\gamma'+1) \Vdash \underset{\sim}{f}(\xi) \in A$$

Now, let $G_\gamma \subseteq P'_\gamma$ be generic over $V[G']$ with $p \restriction_\gamma \in G_\gamma$. Then in $V[G', G_\gamma]$, there exists $A \in [\mathrm{Ord}]^{<\lambda}$ such that–

$$\langle t^p_\gamma, A^p_\gamma \rangle^\frown p \restriction_{(\gamma,\gamma']} {}^\frown q^* \setminus (\gamma'+1) \Vdash \underset{\sim}{f}(\xi) \in A$$

Let $n < \omega$ be such that $\mathrm{lh}\left(t^p_\gamma\right) = n + \mathrm{lh}\left(t^{q^*}_\gamma\right)$. Then $p \restriction_\gamma$ extends $q^{**} \restriction_\gamma$, and thus forces that for every $s \in \left[\underset{\sim}{A}^p_\gamma\right]^n$, there exists $r_s \geq^* q^* \restriction_{(\gamma,\gamma']}$ and a set A_s bounded in λ, such that–

$$\langle t^{q^*}_\gamma {}^\frown s, \underset{\sim}{A}^p_\gamma \setminus \max(s) \rangle^\frown r_s {}^\frown q^* \setminus (\gamma'+1) \Vdash \underset{\sim}{f}(\xi) \in A_s$$

Let r be a $P_{\gamma+1}$-name for the direct extension of q^* which is forced by–

$$\langle t^{q^*}_\gamma {}^\frown s, \underset{\sim}{A}^p_\gamma \setminus \max(s) \rangle$$

to be r_s, for every s of length n. Then $r \geq^* q^* \restriction_{(\gamma,\gamma']}$, and by direct extending r inside the support of q^*, we can assume that $r \geq^* q^{**} \restriction_{(\gamma,\gamma']}$ (note that the coordinates in which a non-direct extension is taken in the extension $r \geq^* q^* \restriction_{(\gamma,\gamma']}$ does not lie inside $\operatorname{supp}(q^*)$).

By taking a union of the sets A_s above, there exists a set of ordinals $A \in V[G', G_\mu]$ with $|A| < \lambda$ such that–

$$\langle t^{q^*}_\gamma, \underset{\sim}{A}^p_\gamma \rangle^\frown r^\frown q^* \setminus (\gamma+1) \Vdash \underset{\sim}{f}(\xi) \in A$$

G_γ was an arbitrary generic set with $p \restriction_\gamma \in G_\gamma$; thus, in $V[G']$,

$$p \restriction_\gamma \Vdash \exists A \in [Ord]^{<\lambda}, \ \langle t^{q^*}_\gamma, \underset{\sim}{A}^p_\gamma \rangle^\frown r^\frown q^* \setminus (\gamma+1) \Vdash \underset{\sim}{f}(\xi) \in A$$

Let $\underset{\sim}{A}$ be a P'_γ-name for the above set A, and let $A^* \in V[G']$ be the set of all possible values of elements in $\underset{\sim}{A}$ as forced by extensions of $p \restriction_\gamma$. Then $A^* \in [\mathrm{Ord}]^{<\lambda}$, and–

$$p \restriction_\gamma {}^\frown \langle t^{q^*}_\gamma, \underset{\sim}{A}^p_\gamma \rangle^\frown r^\frown q^{**} \setminus (\gamma+1) \Vdash \underset{\sim}{f}(\xi) \in A^*$$

This contradicts the minimality of the number of non-direct extensions in the choice of $p \geq q^{**}$. $\square$

Let us mention several immediate corollaries of the last lemma, all of them were introduced in [5]:

Corollary 2.2.7. *Let $\lambda \leq \kappa$ be a regular cardinal and $p \in P_\lambda$. Assume that $\underset{\sim}{\alpha}$ is a P_λ-name for an ordinal. Then there exist $p^* \geq^* p$ and a set of ordinals A of cardinality $|A| < \lambda$, such that $p^* \Vdash \underset{\sim}{\alpha} \in \check{A}$.*

Proof. If λ is a limit of measurables, then it is inaccessible, and then the proof is included in the proof of lemma 2.2.6. Else, let $\lambda' < \lambda$ be the supremum of the set of measurables below λ. Then $P_\lambda = P_{\lambda'}$. We can now repeat the argument in the proof of lemma 2.2.6 for the forcing $P_{\lambda'}$, with minor changes: first define $D = \{r \in P_{\lambda'} \colon \exists A \in [\mathrm{Ord}]^{<\lambda}$ such that $r \Vdash \underset{\sim}{\alpha} \in A\}$. Direct extend $p^* \geq^* p$ and find a club $C \subseteq \lambda'$ such that for every $p^* \leq q \in D$, $q \restriction_{\gamma'+1} {}^\frown p^* \setminus (\gamma'+1) \in D$, where $\gamma' \in C$ is a above the finite set of non-direct extensions taken in the extension $q \geq p^*$. Then, direct extend $p^{**} \geq^* p^*$, without changing the support, as in the previous lemma. Arguing as above, p^{**} has a direct extension which decides $\underset{\sim}{\alpha}$ up to $< \lambda$-many possibilities.

We remark that if $\lambda > {\lambda'}^+$, a simpler argument exists: by GCH, $P_{\lambda'}$ is $\lambda - c.c.$. let $A \in V$ be the set–

$$A = \{\xi \colon \exists q \geq p,\ q \Vdash \xi = \underset{\sim}{\alpha}\}$$

then $|A| < \lambda$ and $p \Vdash \underset{\sim}{\alpha} \in A$ (here a direct extension of p is not required). □

Corollary 2.2.8. *Let $\lambda \leq \kappa$ be inaccessible, and assume that $G_\lambda \subseteq P_\lambda$ is generic over V. Then λ is still regular iv $V[G_\lambda]$. Moreover, every function $f \colon \lambda \to \lambda$ in $V[G_\lambda]$ is dominated by a function $g \colon \lambda \to \lambda$ in V.*

Proof. Assume that λ is singular in $V[G_\lambda]$. Let $\mu = \mathrm{cf}(\lambda)$. Let $f \colon \mu \to \lambda$ be an increasing cofinal sequence in $V[G_\lambda]$. Let $p \in P_\lambda$ be a condition which forces this. We argue that there exists $\delta < \lambda$ and $p^* \geq p$ such that $p^* \Vdash \mathrm{Im}(f) \subseteq \delta$, which is a contradiction. Assume without loss of generality that p is the weakest condition in P_λ.

Work in an arbitrary generic extension of V with the forcing $P_{\mu+1}$. We argue that every condition $q \in P_\lambda \setminus (\mu+1)$ has a direct extension $q^* \in P_\lambda \setminus (\mu+1)$ and function $\alpha \mapsto F(\alpha)$ such that for every $\xi < \mu$, $F(\xi)$ is a bounded subset of λ, and–

$$q^* \Vdash \underset{\sim}{f}(\xi) \in F(\xi)$$

Indeed, given $\xi < \mu$, $\underset{\sim}{f}(\xi)$ is a $P_\lambda \setminus (\mu+1)$-name for an ordinal below λ. By corollary 2.2.7, every $q \in P_\lambda \setminus (\mu+1)$ can be direct extended to $q^* \in P_\lambda \setminus (\mu+1)$ such that for some set of ordinals $A_\xi \subseteq \lambda$ with $|A_\xi| < \lambda$, $q^* \Vdash \underset{\sim}{f}(\xi) \in A_\xi$. Since the direct extension order of $P_\lambda \setminus (\mu+1)$ is more then μ-closed, we can find a single $q^* \in P_\lambda \setminus (\mu+1)$, and, for every $\xi < \mu$, a bounded subset $A_\xi \subseteq \lambda$ such that $q^* \Vdash \forall \xi < \mu,\ \underset{\sim}{f}(\xi) \in A_\xi$; then, set $F(\xi) = A_\xi$ as desired.

Since we worked in an arbitrary generic extension above $(\mu+1)$ and gave a density argument in $P_\lambda \setminus (\mu+1)$, we can assume that there exists $p^* \in G$ such that–

$$p^* \restriction_{\mu+1} \Vdash \text{there exists a function } \xi \mapsto F(\xi) \text{ such that, for every } \xi < \mu,$$
$$F(\xi) \text{ is a bounded subset of } \lambda \text{ and } p^* \setminus (\mu+1) \Vdash \underset{\sim}{f}(\xi) \in F(\xi)$$

Finally, define, in V,

$$\delta = \sup\left(\bigcup_{\xi<\mu} \{\beta < \lambda \colon \exists q \geq p^* \restriction_{\mu+1},\ q \Vdash \check{\beta} \in \underset{\sim}{F}(\xi)\} \right)$$

and note that $\delta < \lambda$ and $p^* \Vdash \operatorname{Im}(\underset{\sim}{f}) \subseteq \delta$.

Let us argue now that every function $f \colon \lambda \to \lambda$ in $V[G]$ is dominated by a function $g \colon \lambda \to \lambda$ in V. First, in $V[G]$, f is dominated by an increasing function $f' \colon \lambda \to \lambda$. By 2.2.6, f' is dominated on a club $C \subseteq \lambda$ by a function $g' \colon \lambda \to \lambda$ in V. Given $\xi < \kappa$, let $c_\xi = \min(C \setminus \xi + 1)$. Finally, define $g \colon \lambda \to \lambda$,

$$g(\xi) = g'(c_\xi)$$

Then for every $\xi < \kappa$, $f(\xi) \leq f'(\xi) \leq f'(c_\xi) < g'(c_\xi) = g(\xi)$. □

Corollary 2.2.9. *Let $\lambda \leq \kappa$ be inaccessible. The forcing P_λ preserves stationary subsets of λ.*

Proof. It suffices to prove that for every club in κ, $C \in V[G]$, there exists a club in κ, $D \in V$, such that $D \subseteq C$. In $V[G]$, let $f \colon \kappa \to \kappa$ be the increasing enumeration of C. By corollary 2.2.8, there exists $g \in V$ which dominates f. Let D be the set of closure points of g. Clearly, D is a club. Let us prove that $D \subseteq C$. Given $\alpha \in D$, α is a closure point of f, and thus a limit point of $\operatorname{Im}(f) = C$. Therefore $\alpha \in C$. □

Recall that a set of ordinals $A \in V[G]$ is called fresh if $A \notin V$ and, for every ordinal $\xi < \sup(A)$, $A \cap \xi \in V$. Every old measurable $\mu < \kappa$ clearly has a fresh unbounded subset: its Prikry sequence.

So if $\sup(A)$ was a measurable cardinal below κ in V, A might be fresh over V. Let us address the case where $\sup(A)$ is κ or κ^+.

Lemma 2.2.10. *$P = P_\kappa$ does not add new unbounded subsets of κ or κ^+ which are fresh over V.*

The proof appears in [8]. Having no fresh subsets of κ, κ^+, together with preservation of cardinals and $2^\kappa = \kappa^+$, leads to the following key property:

Corollary 2.2.11. *Let $W \in V[G]$ be a κ-complete ultrafilter on κ. Then $W \cap V \in V$.*

This follows from proposition 1.0.2 and lemma 2.2.10.

Corollary 2.2.12. *Let $W \in V[G]$ be a normal measure. Then $W \cap V \in V$ is a normal measure of Mitchell order 0 in V.*

Proof. Denote $U = W \cap V$. By corollary 2.2.11, $U \in V$. U inherits normality from W, since it is closed under diagonal intersections. Finally, let us prove that U has Mitchell order 0. Assume otherwise. Then U concentrates on the set Δ of measurables below κ in V. Hence, $\Delta \in W$. However, in $V[G]$, each cardinal in Δ is singular and has cofinality ω, and by normality of W, it cannot concentrate on Δ. □

2.3 Normal Measures in the Generic Extension

Our goal in this section is to prove theorem 2.1.1, namely, that there exists a bijection between normal measures of Mitchell order 0 on κ in V, and normal measures on κ in $V[G]$.

Let $U \in V$ be any normal measure on κ of Mitchell order 0. After forcing an iteration of Prikry forcings, with any standard support, one can define, in the generic extension $V[G]$, a natural filter which extends U: The filter consisting of sets $(\underset{\sim}{A})_G$, where $\underset{\sim}{A}$ is a name for a subset of κ, such that, for some $p \in G$,

$$\{\alpha < \kappa \colon p \Vdash \check{\alpha} \in A\} \in U$$

or simply $j_U(p) \Vdash \check{\alpha} \in j_U(\underset{\sim}{A})$, in M_U.

Forcing with nonstationary support has the advantage, that this filter is actually a normal, κ-complete ultrafilter.

Lemma 2.3.1. *Let U be a normal measure of Mitchell order 0 on κ. Define $U^* \in V[G]$ as follows: For every P_κ-name $\underset{\sim}{A}$ for a subset of κ, $(\underset{\sim}{A})_G \in U^*$ if and only if there exists $p \in G$ such that $j_U(p) \Vdash \check{\kappa} \in j_U(\underset{\sim}{A})$. Then U^* is a normal, κ-complete ultrafilter in $V[G]$, which extends U.*

Proof. Denote for simplicity $P = P_\kappa$. Let us first check that U^* is well defined. Assume that $\underset{\sim}{A}, \underset{\sim}{B}$ are P-names for subsets of κ, and $p \in G$ is a condition such that $p \Vdash \underset{\sim}{A} = \underset{\sim}{B}$. Then $j_U(p) \Vdash j_U(\underset{\sim}{A}) = j_U(\underset{\sim}{B})$, and thus $j_U(p) \Vdash \check{\kappa} \in j_U(\underset{\sim}{A})$ if and only if $j_U(p) \Vdash \check{\kappa} \in j_U(\underset{\sim}{B})$, so $(\underset{\sim}{A})_G \in U^*$ if and only if $(\underset{\sim}{B})_G \in U^*$, as desired.

It's not hard to verify that U^* is a filter. Let us prove that it's κ-complete (thus, in particular, it's an ultrafilter). Assume that $\gamma < \kappa$ and $\langle \underset{\sim}{A}_\beta \colon \beta < \gamma \rangle$ is forced by the weakest condition to be a partition of κ.

For every $\alpha \in (\gamma, \kappa)$, let $e(\alpha) \subseteq P \setminus (\alpha + 1)$ be the following $\leq^*$-dense open subset:

$$e(\alpha) = \{r \in P \setminus (\alpha + 1) : \exists \beta^* < \gamma \;\; r \Vdash \check{\alpha} \in \underset{\sim}{A}_{\beta^*}\}$$

By lemma 2.2.3, there exists $p \in G$ and a club $C \subseteq \kappa$ such that for every $\alpha \in C$,

$$p \restriction_{\alpha+1} \Vdash p \setminus \alpha + 1 \in e(\alpha)$$

C is a club, so $C \in U$, and thus,

$$p^\frown 0_{\underset{\sim}{Q}_\kappa} \Vdash \exists \beta^* < \gamma \quad j_U(p) \setminus (\kappa + 1) \Vdash \check{\kappa} \in j_U(\underset{\sim}{A}_{\beta^*})$$

therefore, for some $q > p$, $q \in G$ and $\beta^* < \gamma$,

$$q \Vdash j_U(p) \setminus \kappa \Vdash \check{\kappa} \in j_U(\underset{\sim}{A}_{\beta^*})$$

so $j_U(q) \Vdash \check{\kappa} \in j_U(\underset{\sim}{A}_{\beta^*})$.

Let us prove normality. Assume that $\underset{\sim}{f}$ is a name for a regressive function from κ to κ. Work in M_U. $j_U(\underset{\sim}{f})$ is forced there to be a regressive function. There exists a $P_{\kappa+1}$-name for a dense open subset D of $j_U(P) \setminus (\kappa + 1)$, consisting of all the conditions which force that $j_U(\underset{\sim}{f})(\kappa) = \beta^*$ for some $\beta^* < \kappa$. Let $\alpha \mapsto e(\alpha)$ be a function in V which represents D in the ultrapower construction. We can assume that for every $\alpha < \kappa$, $e(\alpha)$ is a $P_{\alpha+1}$-name, forced by the weakest condition to be a $\leq^*$ dense-open subset of $P_\kappa \setminus (\alpha + 1)$. Now we apply fusion just as before, and find $p \in G$ such that–

$$p^\frown 0_{\underset{\sim}{Q}_\kappa} \Vdash \exists \beta^* < \kappa \; j_U(p) \setminus (\kappa + 1) \Vdash j_U(\underset{\sim}{f})(\kappa) = \beta^*$$

so for some $q \in G$, $q > p$, and for some $\beta^* < \kappa$,

$$j_U(q) \Vdash j_U(\underset{\sim}{f})(\kappa) = \beta^*$$

Therefore, $\{\xi < \kappa \colon f(\xi) = \beta^*\} \in U^*$ as desired. $\square$

Now, given a normal measure $W \in V[G]$ on κ, denote $U = W \cap V$. By corollary 2.2.11, $U \in V$. Our goal will be to prove that $W = U^*$. We start with the following observation:

Lemma 2.3.2. *Let $W \in V[G]$ be a normal measure on κ. Let $j_W : V[G] \to M[H]$ be the ultrapower embedding. Denote $U = W \cap V$, and define $k: M_U \to M$ as follows: $k([f]_U) = [f]_W$. Then:*

1. *$k: M_U \to M$ is an elementary embedding.*
2. *$k \circ j_U = j_W \restriction_V$.*
3. *$k \restriction_\mu = id$, where μ is the first measurable above κ in M_U. In particular, for every $\eta < \mu$, there exists $f \in V$ such that $[f]_U = [f]_W = \eta$.*
4. *$crit(k) = \mu$.*
5. *$(V_\mu)^M = (V_\mu)^{M_U}$.*

Proof. 1. k is well defined, since, if $[f]_U = [g]_U$, then $\{x < \kappa : f(x) = g(x)\} \in U$, and thus this set belongs to W. So $[f]_W = [g]_W$. Similarly, k respects $\in$. Finally, assume that $\varphi(a_1, \ldots, a_n)$ is a formula and $f_1, \ldots, f_n$ are functions. Then:

$$\begin{aligned} M_U \vDash \varphi([f_1]_U, \ldots [f_n]_U) &\iff \{x < \kappa : V \vDash \varphi(f_1(x), \ldots, f_n(x)\} \in U \\ &\iff \{x < \kappa : V \vDash \varphi(f_1(x), \ldots, f_n(x)\} \in W \\ &\iff M[H] \vDash M \vDash \varphi([f_1]_W, \ldots [f_n]_W) \\ &\iff M \vDash \varphi([f_1]_W, \ldots [f_n]_W) \end{aligned}$$

2. Clear from the definitions.

3. First, let us note that for every $\eta \leq \kappa^+$, $k(\eta) = \eta$, using the canonical function which represents η. Also, $k(\kappa^+) = \kappa^+$ since κ^+ is represented by the successor cardinal function. Thus, $\text{crit}(k) > \kappa^+$.

   Now, assume, for contradiction, that there exists $\eta < \mu$ such that $k(\eta) > \eta$. Take the minimal such η. There exists $g \in V[G]$ such that $[g]_W = \eta$. Let $h: \kappa \to \kappa$ be a function in V such that $[h]_U = \eta$. So–

   $$[g]_W = \eta = [h]_U \leq [h]_W$$

and thus, by changing g on a set which doesn't belong to W, we can assume that, for every $\xi < \kappa$,

$$g(\xi) \leq h(\xi) < \text{ the first measurable above } \xi$$

For every $\xi < \kappa$, let $e(\xi)$ be the $P_{\xi+1}$-name for the following set–

$$e(\xi) = \{r \in P \setminus (\xi + 1) : \exists \alpha \leq h(\xi),\ r \Vdash g(\xi) = \check{\alpha}\}$$

this set is $\leq^*$-dense open, since the direct extension order is more than $h(\xi)$-closed. Apply fusion. There exist $p^* \geq^* p$ and a club $C \subseteq \kappa$ such that, for every $\xi \in C$,

$$p^* \restriction_{\xi+1} \Vdash p^* \setminus (\xi + 1) \in e(\xi)$$

in other words,

$$p^* \restriction_{\xi+1} \Vdash \exists \alpha \leq h(\xi),\ p^* \setminus (\xi + 1) \Vdash g(\xi) = \check{\alpha}$$

Define $F \colon C \to V$ as follows: For every $\xi \in C$, set–

$$F(\xi) = \{\alpha \colon \exists a \in P_{\xi+1} \ \ a \geq p^* \restriction_{\xi+1} \text{ and } a^\frown p^* \setminus (\xi + 1) \Vdash g(\xi) = \check{\alpha}\}$$

Note that for every $\xi \in C$, $p^* \Vdash g(\xi) \in \check{F}(\xi)$, and $|F(\xi)| \leq |\xi|^+$. Also, $C \in U$ and thus $C \in W$. Therefore, in $M[H]$,

$$\eta = [g]_W \in [F]_W = j_W(F)(\kappa) = k\left(j_U(F)(\kappa)\right)$$

but, in M_U, $|j_U(F)(\kappa)| \leq \kappa^+$, which is strictly below the critical point of k. So $\eta \in \mathrm{Im}(k)$, i.e., for some $\alpha \leq \eta$, $\eta = k(\alpha)$. But η was the minimal such that $k(\eta) \neq \eta$, so $\alpha < \eta$ and $\alpha = k(\alpha) = \eta$, a contradiction.

4. It suffices to prove that $k(\mu) \neq \mu$. Since μ is measurable in M_U, it suffices to prove that μ is not measurable in M. Assume otherwise. Then in $V[G]$, $\mathrm{cf}(\mu) = \omega$ would hold. Therefore, in V, $\mathrm{cf}(\mu) \leq \kappa$. By closure under κ-sequences, this is true in M_U as well, a contradiction.

5. It suffices to prove that, for every $\alpha < \mu$, $(V_\alpha)^M = (V_\alpha)^{M_U}$. Indeed,

$$(V_\alpha)^M = k\left((V_\alpha)^{M_U}\right) = k''\left((V_\alpha)^{M_U}\right) = (V_\alpha)^{M_U}$$

□

We now have all the tools necessary for the proof of theorem 2.1.1

Proof of theorem 2.1.1. Assume that $W \in V[G]$ is a normal measure. Denote $U = W \cap V$. Let us use the notations of lemma 2.3.2: Assume that $j_W : V[G] \to M[H]$ is the ultrapower embedding of W, and let $k : M_U \to M$ be such that $j_W \restriction_V = k \circ j_U$.

Let us prove that $W = U^*$. Since both are ultrafilters on κ, it suffices to prove that $U^* \subseteq W$. Assume that $X \in U^*$. Let $\underset{\sim}{X} \in V$ be a P_κ-name such that $(\underset{\sim}{X})_G = X$. There exists $p \in G$ such that $j_U(p) \Vdash \check{\kappa} \in j_U(\underset{\sim}{X})$. By applying k,

$$j_W(p) \Vdash \check{\kappa} \in j_W \restriction_V (\underset{\sim}{X})$$

since $\mathrm{crit}(k) > \kappa$. But $j_W(p) \in j_W(G) = H$. Hence, in $M[H]$,

$$\kappa \in (j_W \restriction_V (\underset{\sim}{X}))_H = j_W((\underset{\sim}{X})_G) = j_W(X)$$

so $X \in W$. □

2.4 The Structure of $j_W \restriction_V$

As usual, let $W \in V[G]$ be a normal measure, and denote $U = W \cap V$. Let $\kappa^* = j_U(\kappa)$. Given $\alpha < \kappa$, recall that $\underset{\sim}{U}^*_\alpha$ is a P_α-name, forced by the weakest condition in P_α to be the normal measure on α used in the Prikry forcing $\underset{\sim}{Q}_\alpha$. Let $\underset{\sim}{\mathcal{U}} = \langle \underset{\sim}{U}_\alpha : \alpha \in \Delta \rangle$ be the sequence of names, such that, for every $\alpha \in \Delta$, $\underset{\sim}{U}_\alpha$ is forced by the weakest condition in P_α to be $\underset{\sim}{U}^*_\alpha \cap V$. Given $G \subseteq P_\kappa$ generic over V, let $\mathcal{U} = \langle U_\alpha : \alpha \in \Delta \rangle$ be the interpretation of the names in $\underset{\sim}{\mathcal{U}}$ with respect to the generic G.

Our goal in this section is to factor $j_W \restriction_V$ to an iterated ultrapower of V, while revealing, simultaneously, more and more information about the generic set $H = j_W(G)$.

By induction, we define for every $\alpha < \kappa^*$ a model M_α, an embedding $j_\alpha : V \to M_\alpha$, a measurable cardinal μ_α in M_α and a measure $U_{\mu_\alpha} \in M_\alpha$ on it. The definition goes by induction on $\alpha < \kappa^*$, such that the sequence of models $\langle M_\alpha : \alpha < \kappa^* \rangle$ is a linear iterated ultrapower of V with direct limit M_{κ^*}. The iteration is continuous, namely, for every $\alpha \leq \kappa^*$ limit, M_α is the direct limit of the models $\langle M_{\alpha'} : \alpha' < \alpha \rangle$.

Given $\alpha < \kappa^*$, we define μ_α to be the least measurable μ in M_α, such that for every $\alpha' < \alpha$, $\mu_{\alpha'} < \mu$, and such that $(\mathrm{cf}(\mu))^V > \kappa$. We will define a measure $U_{\mu_\alpha} \in M_\alpha$ on μ_α. We postpone the definition of U_{μ_α}, but mention only that it will have Mitchell order 0. After U_{μ_α} is defined, we take $M_{\alpha+1} = \mathrm{Ult}(M_\alpha, U_{\mu_\alpha})$ and $j_{\alpha+1} = \left(j_{U_{\mu_\alpha}}\right)^{M_\alpha} \circ j_\alpha$.

Our goal in this section will be to prove the following:

Theorem 2.4.1. *$M = M_{\kappa^*}$, $j_W \restriction_V = j_{\kappa^*}$ and $\kappa^* = j_W(\kappa)$. If $\mathcal{U} \in V$, then both M and $j_W \restriction_V$ are definable classes of V.*

Remark 2.4.2. *Given $\alpha < \kappa^*$, we will prove, in the next section, that every inaccessible λ of M_α above $\bar{\mu} = \sup\{\mu_{\alpha'} : \alpha' < \alpha\}$ satisfies $(cf(\lambda))^V > \kappa$. So whenever μ_α is picked as the least measurable above $\bar{\mu}$ with cofinality $> \kappa$ in V, it is simply the least measurable above $\bar{\mu}$. The proof appears in lemma 2.5.6, and a simpler characterization of $\langle \mu_\alpha : \alpha < \kappa^* \rangle$ appears in corollary 2.5.7. In order to avoid complications in the current section, we chose to provide those results, which involve a detailed study of the iteration $\langle M_\alpha : \alpha \leq \kappa^* \rangle$, in the next section.*

The proof of theorem 2.4.1 goes as follows: By induction on $\alpha \leq \kappa^*$, we define an elementary embedding $k_\alpha : M_\alpha \to M$, as follows:

$$k_\alpha \left(j_\alpha(h) \left(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k} \right) \right) = j_W(h) \left(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k} \right)$$

for $h \in V$, $k < \omega$ and $\alpha_0 < \ldots < \alpha_k < \alpha$.

Note that for $\alpha = 0$, $k_0 = k$ is the embedding defined in lemma 2.3.2. In general, it's not trivial that k_α is a well defined elementary embedding. This will be proved in lemma 2.4.3. We denote $\lambda_\alpha = \text{crit}\,(k_\alpha)$. We will prove that for every $\alpha < \kappa^*$, the following properties hold:

(A) $k_\alpha : M_\alpha \to M$ is an elementary embedding, and $j_W \restriction_V = k_\alpha \circ j_\alpha$.

(B) λ_α is measurable in M_α.

(C) λ_α appears as an element in the Prikry sequence of $k_\alpha(\lambda_\alpha)$ in $M[H]$.

(D) $\lambda_\alpha = \mu_\alpha$.

(E) Let $U_{\mu_\alpha} = \{X \subseteq \mu_\alpha : \mu_\alpha \in k_\alpha(X)\} \cap M_\alpha$. Then $U_{\mu_\alpha} \in M_\alpha$, and is a normal measure of Mitchell order 0 there. Moreover, $j_W(\mathcal{U})(k_\alpha(\mu_\alpha)) = k_\alpha(U_{\mu_\alpha})$, and, if $\mathcal{U} \in V$, then $U_{\mu_\alpha} = j_\alpha(\mathcal{U})(\mu_\alpha)$.

After we prove that properties (A)-(E) above hold for every $\alpha < \kappa^*$, we will show that k_{κ^*} is the identity function.

Let us assume now that the M_β-ultrafilter U_{μ_β} and the embedding $k_\beta\colon M_\beta \to M$ have been defined for every $\beta < \alpha$, such that properties (A)-(E) hold. We first prove that k_α is indeed elementary.

Lemma 2.4.3. *$k_\alpha\colon M_\alpha \to M$ is an elementary embedding, and $j_W \restriction_V = k_\alpha \circ j_\alpha$.*

Proof. We prove only that k_α is a well defined injection (and the rest of elementarity follows similarly). Assume that $a, a' \in M_\alpha$. Let $k < \omega$, $h, h' \in V$ and $\alpha_0 < \ldots < \alpha_k < \alpha$ be such that–

$$a = j_\alpha(h)\,(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k})\ \ ,\ \ a' = j_\alpha(h')\,(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k})$$

If α is limit, let $\alpha' < \alpha$ be high enough such that $\mu_{\alpha'} > \mu_{\alpha_k}$. By induction, $j_W \restriction_V = k_{\alpha'} \circ j_{\alpha'}$, and thus–

$$\begin{aligned} & j_W(h)\,(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k}) = j_W(h')\,(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k}) \\ \iff & j_{\alpha'}(h)\,(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k}) = j_{\alpha'}(h')\,(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k}) \\ \iff & j_\alpha(h)\,(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k}) = j_\alpha(h')\,(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k}) \end{aligned}$$

If $\alpha = \alpha' + 1$ is successor, we can assume that $\alpha_k = \alpha'$, and then–

$$\begin{aligned} & j_W(h)\,(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k}) = j_W(h')\,(\kappa, \mu_{\alpha_0}, \ldots, \mu_{i_k}) \\ \iff & \mu_{\alpha'} \in k_{\alpha'}\left(\{y < \mu_{\alpha'} \colon j_{\alpha'}(h)(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_{k-1}}, y) = j_{\alpha'}(h')(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_{k-1}}, y)\}\right) \\ \iff & \{y < \mu_{\alpha'} \colon j_{\alpha'}(h)(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_{k-1}}, y) = j_{\alpha'}(h')(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_{k-1}}, y)\} \in U_{\mu_{\alpha'}} \\ \iff & j_\alpha(h)\,(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k}) = j_\alpha(h')\,(\kappa, \mu_0, \ldots, \mu_{\alpha_k}) \end{aligned}$$

Finally, we argue that $k_\alpha \circ j_\alpha = j_W \restriction_V$: For each $x \in V$, let $c_x\colon \kappa \to V$ be the function such that for every $\xi < \kappa$, $c_x(\xi) = x$. Then–

$$k_\alpha\left(j_\alpha(x)\right) = k_\alpha\left(j_\alpha(c_x)(\kappa)\right) = j_W(c_x)(\kappa) = j_W(x)$$

□

Since α is fixed from now on, we denote simply $\lambda = \lambda_\alpha = \mathrm{crit}\,(k_\alpha)$. Then λ is a regular uncountable cardinal. Our goal will be to prove that it is measurable in M_α, and moreover, $\lambda = \mu_\alpha$. There are several straightforward limitations on the value of λ:

Claim 2.4.4. $\sup\{\mu_{\alpha'} \colon \alpha' < \alpha\} \leq \lambda \leq \mu_\alpha$.

Proof. By the definition of k_α, for every $\alpha' < \alpha$,

$$k_\alpha(\mu_{\alpha'}) = k_\alpha(j_\alpha(id)(\mu_{\alpha'})) = j_W(id)(\mu_{\alpha'}) = \mu_{\alpha'}$$

Now, if $x < \mu_{\alpha'}$ for some $\alpha' < \alpha$, then $j_{\alpha',\alpha}(x) = x$. Thus, for some $h \in V$, and $\alpha_0 < \ldots < \alpha_k < \alpha'$, $x = j_{\alpha'}(h)(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k})$. Denote $\vec{\mu} = \langle \mu_{\alpha_0}, \ldots, \mu_{\alpha_k} \rangle$. Then–

$$k_\alpha(x) = k_\alpha(j_\alpha(h)(\kappa, \vec{\mu})) = j_W(h)(\kappa, \vec{\mu}) = k_{\alpha'}(j_{\alpha'}(h)(\kappa, \vec{\mu})) = k_{\alpha'}(x) = x$$

where the last equality holds since $x < \mu_{\alpha'}$, and, by induction, $\mathrm{crit}(k_{\alpha'}) = \mu_{\alpha'}$.

This shows that $\mathrm{crit}(k_\alpha) \geq \mu_{\alpha'}$ for every $\alpha' < \alpha$.

For the second inequality, recall that μ_α is measurable in M_α which satisfies $(\mathrm{cf}(\mu_\alpha))^V > \kappa$. If $k_\alpha(\mu_\alpha) = \mu_\alpha$, then, by elementarity, μ_α is measurable in M. Therefore, in $M[H]$, $\mathrm{cf}(\mu_\alpha) = \omega$, and thus in $V[G]$, $\mathrm{cf}(\mu_\alpha) = \omega$. Therefore, in V, $\mathrm{cf}(\mu_\alpha) \leq \kappa$, a contradiction. □

Recall that for every $\beta < \alpha$, μ_β appears as an element in the Prikry sequence added to $k_\beta(\mu_\beta)$ in $M[H]$. Assume that it is the $(n_\beta + 1)$-th element in this Prikry sequence, and has an initial segment t_β of length n_β below it. Note that, by induction, $k_\beta(t_\beta) = t_\beta$.

We now provide a useful way to represent elements in the model M_α.

Definition 2.4.5. *An increasing sequence $\langle \alpha_0, \ldots, \alpha_k \rangle$ of ordinals below κ^* is called **nice** if, for every $0 \leq i \leq k$, there are functions $g_i, t_i, F_i \in V$ such that–*

$$\mu_{\alpha_i} = j_{\alpha_i}(g_i)\left(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_{i-1}}\right)$$

$$t_{\alpha_i} = j_{\alpha_i}(t_i)\left(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_{i-1}}\right)$$

$$U_{\mu_{\alpha_i}} = j_{\alpha_i}(F_i)\left(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_{i-1}}\right)$$

(for $i = 0$, $\mu_{\alpha_0} = j_{\alpha_0}(g_0)(\kappa)$, $t_{\alpha_0} = j_{\alpha_0}(t_0)(\kappa)$ and $U_{\mu_{\alpha_0}} = j_{\alpha_0}(F_0)(\kappa)$).

(We remark that the functions F_i used to represent $U_{\mu_{\alpha_i}}$ will be relevant only in the next section, so the third requirement, that includes them, can be omitted from the definition at the moment). It's not hard to prove that, given a pair of nice sequences, the increasing enumeration of their union is nice.

Lemma 2.4.6. *Every element in M_α has the form–*

$$j_\alpha(h)(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k})$$

for some $k < \omega$, $(k+1)$-ary function $h \in V$ and a nice sequence $\langle \alpha_0, \ldots, \alpha_k \rangle$ of ordinals below α.

Proof. We assume that the lemma holds for every $\alpha' < \alpha$. Let $x \in M_\alpha$.

If α is limit: There exists $\alpha' < \alpha$ and $x' \in M_{\alpha'}$ such that $x = j_{\alpha',\alpha}(x')$. By induction, $x' = j_{\alpha'}(h)(\mu_{\alpha_0}, \dots, \mu_{\alpha_k})$ for a nice sequence $\langle \alpha_0, \dots, \alpha_k \rangle$ below α'. Then $x = j_\alpha(h)(\mu_{\alpha_0}, \dots, \mu_{\alpha_k})$, as desired.

If $\alpha = \alpha' + 1$ is successor: Let $f \in M_{\alpha'}$ be a function such that $x = j_{\alpha',\alpha}(f)(\mu_{\alpha'})$. Let $h_1, h_2, h_3, h_4 \in V$ be functions, and $\langle \alpha_0, \dots, \alpha_k \rangle$, $\langle \beta_0, \dots, \beta_l \rangle$, $\langle \gamma_0, \dots, \gamma_s \rangle$, $\langle \delta_0, \dots, \delta_r \rangle$ be nice sequences below α' such that–

$$f = j_{\alpha'}(h_1)(\mu_{\alpha_0}, \dots, \mu_{\alpha_k}) \ , \ \mu_{\alpha'} = j_{\alpha'}(h_2)(\mu_{\beta_0}, \dots, \mu_{\beta_l})$$

$$t_{\alpha'} = j_{\alpha'}(h_3)(\mu_{\gamma_0}, \dots, \mu_{\gamma_s}) \ , \ U_{\mu_{\alpha'}} = j_{\alpha'}(h_4)(\mu_{\delta_0}, \dots, \mu_{\delta_r})$$

The increasing enumeration of–

$$\langle \alpha_0, \dots, \alpha_k \rangle \cup \langle \beta_0, \dots, \beta_l \rangle \cup \langle \gamma_0, \dots, \gamma_s \rangle \cup \langle \delta_0, \dots, \delta_r \rangle \cup \langle \alpha' \rangle$$

is a nice sequence. Denote it by $\langle \varepsilon_0, \dots, \varepsilon_m, \alpha' \rangle$, where $\varepsilon_m < \alpha'$.

By modifying the function h_1 in V, we can assume for simplicity that–

$$f = j_{\alpha'}(h_1)(\mu_{\varepsilon_0}, \dots, \mu_{\varepsilon_m})$$

Define, in V, a function h, as follows:

$$h(\langle \nu_0, \dots, \nu_m, \nu \rangle) = h_1(\nu_0, \dots, \nu_m)(\nu)$$

Then $j_\alpha(h)(\mu_{\varepsilon_0}, \dots, \mu_{\varepsilon_m}, \mu_{\alpha'}) = x$. □

We now introduce several notations. We fix those notations throughout the proof that properties (A)-(E) hold at α. Recall that crit(k_α) is denoted by λ. Let $h \in V$ be a function such that–

$$\lambda = j_\alpha(h)(\kappa, \mu_{\alpha_0}, \dots, \mu_{\alpha_k})$$

for a nice sequence $\langle \alpha_0, \dots, \alpha_k \rangle$ below α. Fix, for every $0 \le i \le k$, functions $g_i, t_i \in V$ as in the definition of a nice sequence. In other words–

$$\mu_{\alpha_i} = j_{\alpha_i}(g_i)\left(\kappa, \mu_{\alpha_0}, \dots, \mu_{\alpha_{i-1}}\right)$$

$$t_{\alpha_i} = j_{\alpha_i}(t_i)\left(\kappa, \mu_{\alpha_0}, \dots, \mu_{\alpha_{i-1}}\right)$$

Remark 2.4.7. *1. The functions g_i might be more or less the same. For instance, set, for every $\xi < \kappa$, $g_0 = s(\xi) =$ the first measurable in V strictly above ξ, and $g_1(\xi, \nu) = s(\xi)$. Then $\mu_0 = j_0(g_0)(\kappa)$ and $\mu_1 = j_1(g_0)(\kappa) = j_1(g_1)(\kappa)$.*

2. It is not necessarily true that, given $\xi, \vec{\nu}$, $h(\xi, \vec{\nu}) \geq g_i(\xi, \nu_0, \ldots, \nu_{i-1})$. For instance, take, μ_{α_k} to be a measurable of Mitchell order > 0 in M_U, and λ to be the first measurable above it in $M_{\mu_{\alpha_k}+1} = Ult\left(M_{\alpha_k}, U_{\mu_{\alpha_k}}\right)$. Then $\lambda = j_{\alpha_k+1}(h)(\kappa, \mu_{\alpha_k})$, where $h(\xi, \nu) = s(\nu)$. Assume that $\mu_{\alpha_k} = j_U(f)(\xi)$ for some $f \in V$. In $M[H]$, $k_{\mu_{\alpha_k}+1}(\lambda) < k_{\mu_{\alpha_k}}(\mu_{\alpha_k})$, namely, $h(\xi, \mu_{\alpha_k}(\xi)) < f(\xi)$ for a set of ξ-s in W, where $\xi \mapsto \mu_{\alpha_k}(\xi)$ is a function in $V[G]$ represents μ_{α_k} in the ultrapower with W.

Given $\beta < \alpha$, recall that, by induction, μ_β appears in the Prikry sequence of $k_\beta(\mu_\beta)$. For every $0 \leq i \leq k$, denote by $n_i < \omega$ the length of the finite sequence t_i, which is the initial segment of the Prikry sequence of $k_{\alpha_i}(\mu_{\alpha_i})$ below μ_{α_i}. Then μ_{α_i} is the (n_i+1)-th element in this Prikry sequence.

For every $i \leq k$, we define, in $V[G]$, a function $\xi \mapsto \mu_{\alpha_i}(\xi)$ such that $[\xi \mapsto \mu_{\alpha_i}(\xi)]_W = \mu_{\alpha_i}$:

- For $i = 0$, set the (n_0+1)-th element in the Prikry sequence of $g_0(\xi)$ to be $\mu_{\alpha_0}(\xi)$.
- Assume that $i \leq k$, and the functions $\xi \mapsto \mu_{\alpha_j}(\xi)$ have been defined for every $j < i$. Let $\mu_{\alpha_i}(\xi)$ be the (n_i+1)-th element in the Prikry sequence of $g_i\left(\mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_{i-1}}(\xi)\right)$.

For every $0 \leq i \leq k$, $[\xi \mapsto \mu_{\alpha_i}(\xi)]_W = \mu_{\alpha_i}$, and–

$$t_{\alpha_i} = \left[\xi \mapsto t_i\left(\xi, \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_{i-1}}(\xi)\right)\right]_W$$

where the last equality follows since $\operatorname{crit}(k_{\alpha_i}) = \mu_{\alpha_i}$ and thus $k_{\alpha_i}(t_{\alpha_i}) = t_{\alpha_i}$.

We fix an abbreviation, $\xi \mapsto \vec{\mu}(\xi)$ for the function $\xi \mapsto \langle \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_k}(\xi) \rangle$. Given $\xi, \vec{\nu} = \langle \nu_0, \ldots, \nu_k \rangle$, denote–

$$\vec{t}(\xi, \vec{\nu}) = \langle t_0(\xi), t_1(\xi, \nu_0), \ldots, t_k(\xi, \nu_0, \ldots, \nu_{k-1}) \rangle$$

Our next goal is lemma 2.4.11, which generalizes the Fusion Lemma 2.2.3. We deal there with sets which are $\leq^*$ dense open above conditions which decide the values of $\langle \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_k}(\xi) \rangle$. We first define the notion of a C-tree, which consists of sequences $\langle \xi, \vec{\nu} \rangle = \langle \xi, \nu_0, \ldots, \nu_k \rangle$ which are possible candidates for the exact values of $\langle \xi, \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_k}(\xi) \rangle$. Then, we define in 2.4.10 whenever such a candidate is admissible for a given condition $p \in G$, in the sense that p can be extended to force that $\vec{\mu}(\xi) = \vec{\nu}$.

Definition 2.4.8. *A tree $T \subseteq [\kappa]^{k+1}$ is called a C-tree (with respect to a fixed nice sequence $\langle \alpha_0, \ldots, \alpha_k \rangle$) if $Succ_T(\langle\rangle)$ is a club in κ, and for every $i < k$ and $\langle \xi, \nu_0, \ldots, \nu_i \rangle \in T$, $Succ_T(\xi, \nu_0, \ldots, \nu_i)$ is a club in $g_{i+1}(\xi, \nu_0, \ldots, \nu_i)$.*

Given $i < k$ and a sequence $\langle \xi, \nu_0, \ldots, \nu_i \rangle$, a C-tree above it is a tree $T \subseteq [\kappa]^{k-i}$, such that $Succ_T(\langle\rangle)$ is a club in $g_{i+1}(\xi, \nu_0, \ldots, \nu_i)$ and, for every $i+1 \leq j \leq k-1$ and $\langle \nu_{i+1}, \ldots, \nu_j \rangle \in T$, $Succ_T(\nu_{i+1}, \ldots, \nu_j)$ is a club in $g_{j+1}(\xi, \nu_0, \ldots, \nu_j)$.

Claim 2.4.9. *Let T be a C-tree. Then, in $V[G]$,*

$$\{\xi < \kappa \colon \langle \xi, \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_k}(\xi) \rangle \in T\} \in W$$

Proof. Work in $V[G]$. First, $\{\xi < \kappa \colon \mu_{\alpha_0}(\xi) \in \mathrm{Succ}_T(\xi)\} \in W$. Indeed, for each $\xi \in \mathrm{Succ}_T(\langle\rangle) \in W$, $\mathrm{Succ}_T(\xi)$ is a club in $g_0(\xi)$, and thus–

$$\mu_0 \in k_0\left([\xi \mapsto \mathrm{Succ}_T(\langle \xi \rangle)]_U\right)$$

This holds since $[\xi \mapsto \mathrm{Succ}_T(\langle \xi \rangle)]_U$ is a club in $[g_0]_U = \mu_{\alpha_0}$ and thus belongs to $U_{\mu_{\alpha_0}}$.

Now proceed by induction. For every $i \leq k-1$,

$$\{\xi < \kappa \colon \mu_{\alpha_{i+1}}(\xi) \in \mathrm{Succ}_T(\langle \xi, \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_i}(\xi) \rangle)\} \in W$$

Indeed, denote–

$$C = j_{\alpha_{i+1}}\left(\langle \xi, \nu_0, \ldots, \nu_i \rangle \mapsto \mathrm{Succ}_T(\xi, \nu_0, \ldots, \nu_i)\right)(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_i})$$

Then C is a club in $j_{\alpha_{i+1}}(g_{i+1})(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_i}) = \mu_{\alpha_{i+1}}$. Thus $C \in U_{\mu_{\alpha_{i+1}}}$, and $\mu_{\alpha_{i+1}} \in k_{\alpha_{i+1}}(C)$, as desired. □

Definition 2.4.10. *Fix $\alpha < \kappa$ and a nice sequence $\langle \alpha_0, \ldots, \alpha_k \rangle$ below α. Let $p \in P_\kappa$ be a condition and $\langle \xi, \nu_0, \ldots, \nu_k \rangle$ be a sequence below κ. Let us define whenever $\langle \xi, \nu_0, \ldots, \nu_k \rangle$ is admissible for p, and in that case, define as well an extension $p^\frown\langle \xi, \nu_0, \ldots, \nu_k \rangle \geq p$ in P_κ.*

1. *$\langle \xi, \nu_0 \rangle$ is admissible for p if–*

$$p \restriction_{g_0(\xi)} \Vdash \langle t_0(\xi)^\frown \langle \nu_0 \rangle, A^p_{g_0(\xi)} \setminus (\nu_0 + 1) \rangle \text{ is compatible with } p(g_0(\xi))$$

if this holds, and $t^p_{g_0(\xi)}$ is an initial segment of $t_0(\xi)^\frown\langle \nu_0 \rangle$, let–

$$p^\frown\langle \xi, \nu_0 \rangle = p \restriction_{g_0(\xi)}{}^\frown \langle t_0(\xi)^\frown \langle \nu_0 \rangle, A^p_{g_0(\xi)} \setminus (\nu_0 + 1) \rangle^\frown p \setminus (g_0(\xi) + 1)$$

otherwise, let $p^\frown\langle \xi, \nu_0 \rangle = p$.

2. *Let* $0 \leq m < k$. *Assume that* $\langle \xi, \nu_0, \ldots, \nu_m \rangle$ *is admissible for* p *and* $p^\frown \langle \xi, \nu_0, \ldots, \nu_m \rangle$ *has been defined. Denote–*

$$g_{m+1} = g_{m+1}\left(\xi, \nu_0, \ldots, \nu_m\right)$$

$$t_{m+1} = t_{m+1}\left(\xi, \nu_0, \ldots, \nu_m\right)$$

We say that $\langle \xi, \nu_0, \ldots, \nu_{m+1} \rangle$ *is admissible for* p *if–*

$$p^\frown \langle \xi, \nu_0, \ldots, \nu_m \rangle \restriction_{g_{m+1}} \Vdash \langle t_{m+1}{}^\frown \langle \nu_{m+1} \rangle, A^p_{g_{m+1}} \setminus (\nu_{m+1}+1) \rangle \text{ is compatible with } \left(p^\frown \langle \xi, \nu_0, \ldots, \nu_m \rangle\right)(g_{m+1})$$

if this holds, and $t^{p^\frown \langle \xi, \nu_0, \ldots, \nu_m \rangle}_{g_{m+1}}$ *is an initial segment of* $t_{m+1}{}^\frown \langle \nu_{m+1} \rangle$, *let–*

$$\begin{aligned} p^\frown \langle \xi, \nu_0, \ldots, \nu_{m+1} \rangle = & \left(p^\frown \langle \xi, \nu_0, \ldots, \nu_m \rangle \restriction_{g_{m+1}}\right)^\frown \\ & \langle t_{m+1}{}^\frown \langle \nu_{m+1} \rangle, A^p_{g_{m+1}} \setminus (\nu_{m+1}+1) \rangle^\frown \\ & \left(p^\frown \langle \xi, \nu_0, \ldots, \nu_m \rangle\right) \setminus (g_{m+1}+1) \end{aligned}$$

else, set $p^\frown \langle \xi, \nu_0, \ldots, \nu_{m+1} \rangle = p^\frown \langle \xi, \nu_0, \ldots, \nu_m \rangle$.

Finally, assume that p *is a condition,* $\xi < \kappa$, $i < k$ *and* $\langle \nu_0, \ldots, \nu_i \rangle$ *is a sequence such that* $p \Vdash \langle \underset{\sim}{\mu}_{\alpha_0}(\xi), \ldots, \underset{\sim}{\mu}_{\alpha_i}(\xi) \rangle = \langle \nu_0, \ldots, \nu_i \rangle$. *Given a sequence* $\langle \nu_{i+1}, \ldots, \nu_k \rangle$, *we can define similarly whether it is admissible for* p*; if it is, we say that* $\langle \nu_{i+1}, \ldots, \nu_k \rangle$ *is admissible for* p *above* $\langle \xi, \nu_0, \ldots, \nu_i \rangle$, *and define, in a similar way as above, the condition* $p^\frown \langle \nu_{i+1}, \ldots, \nu_k \rangle$.

Lemma 2.4.11 (Multivariable Fusion)**.** *Fix* $\alpha < \kappa$ *and a nice sequence* $\langle \alpha_0, \ldots, \alpha_k \rangle$ *below* α. *Let* $p \in P_\kappa$ *be a condition. Assume that for every* $\langle \xi, \nu_0, \ldots, \nu_k \rangle$ *below* κ *there exists a subset* $e\left(\xi, \vec{\nu}\right) \subseteq P_\kappa \setminus (\nu_k + 1)$ *which is* $\leq^*$*-dense open above every condition* $q \in P_\kappa \setminus (\nu_k+1)$ *which forces that* $\underset{\sim}{\vec{\mu}}(\xi) = \vec{\nu}$. *Then there exists* $p^* \geq^* p$ *and a* C*-tree* T, *such that for every* $\langle \xi, \nu_0, \ldots, \nu_k \rangle \in T$ *which is admissible for* p^*,

$$\left(p^{*\frown} \langle \xi, \vec{\nu} \rangle\right) \restriction_{\nu_k+1} \Vdash \left(p^{*\frown} \langle \xi, \vec{\nu} \rangle\right) \setminus (\nu_k+1) \in e\left(\xi, \vec{\nu}\right)$$

Proof. For every $i < k$ and $\langle \xi, \nu_0, \ldots, \nu_i \rangle$, we define a subset $e\left(\xi, \nu_0, \ldots, \nu_i\right) \subseteq P \setminus (\nu_i + 1)$ which is $\leq^*$-dense open above every condition $q \in P \setminus (\nu_i + 1)$ which forces that–

$$\langle \underset{\sim}{\mu}_{\alpha_0}(\xi), \ldots, \underset{\sim}{\mu}_{\alpha_i}(\xi) \rangle = \langle \nu_0, \ldots, \nu_i \rangle$$

as follows:

$$
\begin{aligned}
e(\xi,\nu_0,\dots,\nu_i) = \{ & q\in P\setminus(\nu_i+1) : \text{there exists a C-tree } T \text{ above } \langle \xi,\nu_0,\dots,\nu_i\rangle \\
& \text{such that, for every } \langle \nu_{i+1},\dots,\nu_k\rangle\in T, \text{ which is admissible for} \\
& q \text{ above } \langle \xi,\nu_0,\dots,\nu_i\rangle, \\
& (q^\frown\langle\nu_{i+1},\dots,\nu_k\rangle)\restriction_{\nu_k+1}\Vdash (q^\frown\langle\nu_{i+1},\dots,\nu_k\rangle)\setminus(\nu_k+1)\in e(\xi,\vec{\nu})\}
\end{aligned}
$$

The lemma now follows by applying, repeatedly, the following claim:

Claim 2.4.12. *Let $0\le i<k$ and fix an increasing sequence $\langle\xi,\nu_0,\dots,\nu_i,\nu_{i+1}\rangle$. Assume that $e(\xi,\nu_0,\dots,\nu_i,\nu_{i+1})$ is $\le^*$-dense open above every condition in $P\setminus(\nu_{i+1}+1)$ which forces that $\langle \underset{\sim}{\mu}_{\alpha_0}(\xi),\dots,\underset{\sim}{\mu}_{\alpha_{i+1}}(\xi)\rangle=\langle\nu_0,\dots,\nu_{i+1}\rangle$. Then $e(\xi,\nu_0,\dots,\nu_i)$ is $\le^*$-dense open above every condition in $P\setminus(\nu_i+1)$ which forces that $\langle \underset{\sim}{\mu}_{\alpha_0}(\xi),\dots,\underset{\sim}{\mu}_{\alpha_i}(\xi)\rangle=\langle\nu_0,\dots,\nu_i\rangle$.*

Proof. Let $p\in P\setminus(\nu_i+1)$ be a condition which forces that $\langle \underset{\sim}{\mu}_{\alpha_0}(\xi),\dots,\underset{\sim}{\mu}_{\alpha_i}(\xi)\rangle=\langle\nu_0,\dots,\nu_i\rangle$. Denote for simplicity $g_{i+1}=g_{i+1}(\xi,\nu_0,\dots,\nu_i)$. First, direct extend $p\restriction_{g_{i+1}}$ such that it decides the length of $t^p_{g_{i+1}}$, and whether $t^p_{g_{i+1}}, t_{i+1}(\xi,\nu_0,\dots,\nu_i)$ are compatible:

1. If $p\restriction_{g_{i+1}}$ decides that $t^p_{g_{i+1}}$ and $t_{i+1}(\xi,\nu_0,\dots,\nu_i)$ are incompatible, do nothing.
2. If $p\restriction_{g_{i+1}}$ decides that the length of $t^p_{g_{i+1}}$ is above $n_{i+1}+2$, direct extend it further, such that for some $\gamma<g_{i+1}$, $p\restriction_{g_{i+1}}\Vdash t^p_{g_{i+1}}(n_{i+1}+1)<\gamma$ (namely, γ bounds the $(n_{i+1}+1)$-th element in the Prikry sequence of g_{i+1}).
3. If $p\restriction_{g_{i+1}}$ decides that $g_{i+1}\notin\operatorname{supp}(p)$, direct extend p such that $t^p_{g_{i+1}}=t_{i+1}(\xi,\nu_0,\dots,\nu_i)$.
4. If $p\restriction_{g_{i+1}}$ decides that the length of $t^p_{g_{i+1}}$ is less or equal than n_{i+1}, direct extend by shrinking $\underset{\sim}{A}^p_{g_{i+1}}$ to $\underset{\sim}{A}^p_{g_{i+1}}\setminus(\max(t_{i+1}(\xi,\nu_0,\dots,\nu_i))+1)$.

Assume that p is already direct extended as described above. Let us direct extend $p^*\restriction_{g_{i+1}}\ge^* p\restriction_{g_{i+1}}$ using the Fusion lemma in the forcing $P\restriction_{(\nu_i,g_{i+1})}$. For every $\nu\in(\nu_i,g_{i+1})$, consider the following

$\leq^*$-dense open subset of $P\restriction_{(\nu+1,g_{i+1})}$:

$$\begin{aligned}E(\nu)=\{&r\in P\restriction_{(\nu+1,g_{i+1})}:\text{ if } r\Vdash t^r_{g_{i+1}}=t_{i+1}(\xi,\nu_0,\ldots,\nu_i)\text{ and }\nu\in \underset{\sim}{A}^r_{g_{i+1}},\\ &\text{there exists a direct extension–}\\ &q=q(\nu)\geq^*\langle t^p_{g_{i+1}}{}^\frown\langle\nu\rangle,\underset{\sim}{A}^r_{g_{i+1}}\setminus(\nu+1)\rangle^\frown p\setminus(g_{i+1}+1)\\ &\text{such that } r^\frown q\in e(\xi,\nu_0,\ldots,\nu_i,\nu)\}\end{aligned}$$

The $\leq^*$-density of $E(\nu)$ follows from the $\leq^*$-density of $e(\xi,\nu_0,\ldots,\nu_i,\nu)$ above any condition which forces that $\langle \underset{\sim}{\nu}_{\alpha_0}(\xi),\ldots,\underset{\sim}{\mu}_{\alpha_{i+1}}(\xi)\rangle=\langle\xi,\nu_0,\ldots,\nu_i,\nu\rangle$.
Apply Fusion, and let $p^*\restriction_{g_{i+1}}\geq^* p\restriction_{g_{i+1}}$ be a direct extension, such that for some club $C=C(\xi,\nu_0,\ldots,\nu_i)\subseteq g_{i+1}$, and for every $\nu\in C$,

$$p^*\restriction_{\nu+1}\Vdash p^*\setminus(\nu+1)\in e(\xi,\nu_0,\ldots,\nu_i,\nu)$$

Shrink C such that $C\cap(\gamma+1)=\emptyset$ (if necessary, namely, if γ was defined and C contains ordinals below it).
Let us define now $p^*(g_{i+1})$. For every $\nu\in C$, such that $p^*\restriction_{g_{i+1}}\Vdash \underset{\sim}{t}^{p^*}_{g_{i+1}}=t_{i+1}(\xi,\nu_0,\ldots,\nu_i)$ and $\nu\in\underset{\sim}{A}^{p^*}_{g_{i+1}}$, let $q(\nu)$ be the condition as in the definition of $E(\nu)$. For every other value of ν, let $q(\nu)=p\setminus g_{i+1}$. Now, direct extend $p(g_{i+1})$ to–

$$p^*(g_{i+1})=\langle \underset{\sim}{t}^{\,p}_{g_{i+1}},\underset{\sim}{A}^p_{g_{i+1}}\cap\left(\triangle_{\nu<g_{i+1}}\underset{\sim}{A}^{q(\nu)}_{g_{i+1}}\right)\cap C\rangle$$

Finally, we define $p^*\setminus(g_{i+1}+1)=q(\nu)$, where ν is the $(n_{i+1}+1)$-th element in the Prikry sequence of g_{i+1}.

Let us argue now that $p^*\in e(\xi,\nu_0,\ldots,\nu_i)$. We first define a C-tree T above $\langle\xi,\nu_0,\ldots,\nu_i\rangle$. Let $\mathrm{Succ}_T(\langle\rangle)=C=C(\xi,\nu_0,\ldots,\nu_i)$. Fix $\nu_{i+1}=\nu\in C$, and let us define $T_{\langle\nu\rangle}$, which is the tree T above the node $\langle\nu\rangle$.

If $p^*\restriction_{g_{i+1}}$ forces that $t^{p^*}_{g_{i+1}}\neq t_{i+1}(\xi,\nu_0,\ldots,\nu_i)$ or $\nu\notin\underset{\sim}{A}^{p^*}_{g_{i+1}}$, let $T_{\langle\nu\rangle}$ be any C-tree above $\langle\xi,\nu_0,\ldots,\nu_i,\nu\rangle$ (we will prove that any branch starting from ν in T is not admissible for p^*).

Else, note that–

$$p^{*\frown}\langle\nu\rangle=p^*\restriction_{g_{i+1}}{}^\frown\langle t^{p^*}_{g_{i+1}}{}^\frown\langle\nu\rangle,A^{p^*}_{g_{i+1}}\setminus(\nu+1)\rangle^\frown p^*\setminus(g_{i+1}+1)\geq^* p^*\restriction_{g_{i+1}}{}^\frown q(\nu)$$

since $\underset{\sim}{A}^{p^*}_{g_{i+1}}\setminus(\nu_{i+1}+1)\subseteq\underset{\sim}{A}^{q(\nu_{i+1})}_{g_{i+1}}$. Thus, $p^{*\frown}\langle\nu\rangle$ belongs to $e(\xi,\nu_0,\ldots,\nu_i,\nu)$. This is witnessed by a $P\restriction_{\nu+1}$-name for a C-tree $\underset{\sim}{T}(\nu)$ above $\langle\xi,\nu_0,\ldots,\nu_i,\nu\rangle$. We construct $T_{\langle\nu\rangle}$ in $V^{P_{\nu+1}}$ to be a C-tree which is forced, by $p^{*\frown}\langle\nu\rangle \restriction_{\nu+1}= p^* \restriction_{\nu+1}$ to be contained in $\underset{\sim}{T}(\nu)$. The definition is inductive: First, let $\mathrm{Succ}_T(\nu) \subseteq g_{i+1}$ be a club in $V^{P_{\nu_i+1}}$ which is forced by $p^* \restriction_{\nu+1}= (p^{*\frown}\langle\nu\rangle) \restriction_{\nu+1}$ to be contained in $\mathrm{Succ}_{\underset{\sim}{T}(\nu)}(\langle\rangle)$; Such a club exists since the forcing $P \restriction_{(\nu_i,\nu+1)}$ has cardinality strictly below g_{i+1}. Now, given $\nu_{i+2} \in \mathrm{Succ}_T(\nu)$, let $\mathrm{Succ}_T(\nu,\nu_{i+2}) \subseteq g_{i+2}(\xi,\nu_0,\ldots,\nu_i,\nu,\nu_{i+2})$ be a club which is forced by $p^* \restriction_{\nu+1}$ to be contained in $\mathrm{Succ}_{\underset{\sim}{T}(\nu)}(\langle\nu_{i+2}\rangle)$. Continue in this fashion.

This finishes the definition of T. Finally, assume that $\langle\nu_{i+1},\ldots,\nu_k\rangle$ belongs to T and is admissible for p^* above $\langle\xi,\nu_0,\ldots,\nu_i\rangle$. Then $p^* \restriction_{\nu_{i+1}+1}$ forces that $\langle\nu_{i+2},\ldots,\nu_k\rangle \in \underset{\sim}{T}(\nu)$. By admissibility of $\langle\nu_{i+1},\ldots,\nu_k\rangle$ for p^*, $\nu_{i+1} \in \underset{\sim}{A}^{p^*}_{g_{i+1}}$, and $t^{p^*}_{g_{i+1}}$ is compatible with, but not a strict initial segment of $t_{i+1}(\xi,\nu_0,\ldots,\nu_i)$. Since ν_{i+1} belongs to $C(\xi,\nu_0,\ldots,\nu_i)$, and in particular is above γ, $t^{p^*}_{g_{i+1}} = t_{i+1}(\xi,\nu_0,\ldots,\nu_i)$. Thus,

$$p^*(g_{i+1}) = \langle t_{i+1}(\xi,\nu_0,\ldots,\nu_i), \underset{\sim}{A}^{p^*}_{g_{i+1}}\rangle$$

and as before, $p^{*\frown}\langle\nu_{i+1}\rangle \geq^* q(\nu_{i+1})$. Thus $p^{*\frown}\langle\nu_{i+1}\rangle$ forces that $\langle\nu_{i+2},\ldots,\nu_k\rangle \in \underset{\sim}{T}(\nu_{i+1})$ and therefore,

$$(p^{*\frown}\langle\nu_{i+1},\nu_{i+2},\ldots,\nu_k\rangle) \restriction_{\nu_k+1} \Vdash (p^{*\frown}\langle\nu_{i+1},\nu_{i+2},\ldots,\nu_k\rangle) \setminus (\nu_k+1) \in e(\xi,\nu_0,\ldots,\nu_k)$$

as desired. $\square$

Let us prove that the above claim completes the proof of the Multivariable Fusion Lemma. Let $\xi < \kappa$. By applying the claim repeatedly, the set $e(\xi)$ is $\leq^*$-dense open, where $e(\xi)$ is defined as follows:

$$\begin{aligned} e(\xi) =\{q \in P \setminus (\xi+1) : &\text{ there exists a C-tree } T \text{ above } \langle\xi\rangle \text{ such that, for every} \\ &\langle\nu_0,\ldots,\nu_k\rangle \in T, \text{ which is admissible for } q \text{ above } \langle\xi\rangle, \\ &(q^\frown\langle\nu_0,\ldots,\nu_k\rangle) \restriction_{\nu_k+1} \Vdash (q^\frown\langle\nu_0,\ldots,\nu_k\rangle) \setminus (\nu_k+1) \in e(\xi,\vec{\nu})\} \end{aligned}$$

Thus, given a condition $p \in P_\kappa$, there exists $p^* \geq^* p$ and a club $C \subseteq \kappa$, such that, for every $\xi \in C$,

$$p^* \restriction_{\xi+1} \Vdash p^* \setminus (\xi+1) \in e(\xi)$$

In particular, $p^* \restriction_{\xi+1}$ forces that there exists a $P_{\xi+1}$-name for a C-tree $\underset{\sim}{T}(\xi)$ above $\langle\xi\rangle$, such that for every $\langle\nu_0,\ldots,\nu_k\rangle \in \underset{\sim}{T}(\xi)$ which is admissible for $p^* \setminus (\xi+1)$ above ξ,

$$(p^* \setminus (\xi+1))^\frown \langle\nu_0,\ldots,\nu_k\rangle \restriction_{\nu_k+1} \Vdash (p^* \setminus (\xi+1))^\frown \langle\nu_0,\ldots,\nu_k\rangle \setminus (\nu_k+1) \in e(\xi,\nu_0,\ldots,\nu_k)$$

Now, we can construct in V the C-tree T as desired in the formulation of the lemma, such that $\mathrm{Succ}_T(\langle\rangle) = C$, and, for every $\xi \in C$, $T_{\langle\xi\rangle}$ is a tree in V which is forced by $p^* \restriction_{\xi+1}$ to be contained in $\underset{\sim}{T}(\xi)$. Then p^*, T are as desired. $\square$

Remark 2.4.13. *The condition p^* and the C-tree T, obtained from the Multivariable Fusion Lemma, can be assumed to satisfy the following property: For every $i < k$, $\langle \xi, \nu_0, \ldots, \nu_i \rangle \in T$ which is admissible for p^* , and for every $\nu_{i+1} \in Succ_T(\xi, \nu_0, \ldots, \nu_i)$,*

$$p^{*\frown}\langle \xi, \nu_0, \ldots, \nu_i \rangle \restriction_{g_{i+1}(\xi, \nu_0, \ldots, \nu_i)} \| \ \langle \xi, \nu_0, \ldots, \nu_i, \nu_{i+1} \rangle \text{ is admissible for } p^*$$

this requires a minor change in the definition of the set $e(\xi, \nu_0, \ldots, \nu_i)$, which is adding the above as requirement (the same proof provided shows that this additional requirement holds).

Thus, if we apply the standard density argument and choose the condition p^* provided by the Multivariable Fusion Lemma inside G, it follows that–

$$\{\xi < \kappa \colon \langle \xi, \vec{\mu}(\xi) \rangle \in T \text{ is admissible for } p^* \text{ and } p^{*\frown}\langle \xi, \vec{\mu}(\xi) \rangle \in G\} \in W$$

Indeed, note first that $X = \{\xi < \kappa \colon \langle \xi, \vec{\mu}(\xi) \rangle \in T\} \in W$ by claim 2.4.9. Note that if $Y = \{\xi \in X \colon \langle \xi, \vec{\mu}(\xi) \rangle$ is admissible for $p^*\} \in W$ then $\{\xi \in Y \colon p^{*\frown}\langle \xi, \vec{\mu}(\xi) \rangle \in G\} \in W$, since $p^* \in G$, by the definition of the functions $\vec{\mu}(\xi)$.

Thus, it's enough to argue that–

$$\{\xi < \kappa \colon \langle \xi, \vec{\mu}(\xi) \rangle \in T \text{ is admissible for } p^*\} \in W$$

Indeed, we proceed by induction on $i \leq k$. Assume that–

$$\{\xi \in X \colon p^{*\frown}\langle \xi, \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_i}(\xi) \rangle \text{ is admissible for } p^*\} \in W$$

For every such $\xi < \kappa$,

$$p^{*\frown}\langle \xi, \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_i}(\xi) \rangle \restriction_{g_{i+1}(\xi, \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_i}(\xi))} \| \ \langle \xi, \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_{i+1}}(\xi) \rangle \text{ is admissible for } p^*$$

and the decision must be positive for a set of ξ-s in W, since $t_{i+1}(\mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_i}(\xi))^\frown\langle \mu_{\alpha_{i+1}}(\xi) \rangle$ is an initial segment of $g_{i+1}(\xi, \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_i}(\xi))$ for a set of ξ-s in W. Therefore,

$$\{\xi \in X \colon p^{*\frown}\langle \xi, \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_{i+1}}(\xi) \rangle \text{ is admissible for } p^*\} \in W$$

We are now ready to prove that λ is measurable in M_α, which is property (B) above.

Lemma 2.4.14. *λ is measurable in M_α.*

Proof. Assume otherwise. Then it can be assumed that for every ξ and $\vec{\nu}$, $h(\xi,\vec{\nu})$ is a non-measurable regular cardinal. Let $f \in V[G]$ be a function such that $[f]_W = \lambda$. Let $\underset{\sim}{f} \in V$ be a P_κ-name such that $(\underset{\sim}{f})_G = f$. Similarly, let $\underset{\sim}{\vec{\mu}} \in V$ be the sequence of P–names $\langle \underset{\sim}{\mu}_{\alpha_0}(\xi), \dots, \underset{\sim}{\mu}_{\alpha_k}(\xi)\rangle$ described above. In $M[H]$,

$$[f]_W < j_W(h)(\kappa, \vec{\mu})$$

and thus we can assume that there exists a condition $p \in G$ such that, for every $\xi < \kappa$,

$$p \Vdash \underset{\sim}{f}(\xi) < h(\xi, \underset{\sim}{\vec{\mu}}(\xi))$$

From now on we work above p. We can also assume that p forces, for every $0 \le i \le k$, that $\underset{\sim}{\mu}_{\alpha_i}(\xi)$ is the $n_i + 1$-th element in the Prikry sequence of $g_i\left(\xi, \underset{\sim}{\mu}_{\alpha_0}(\xi), \dots, \underset{\sim}{\mu}_{\alpha_{i-1}}(\xi)\right)$.

Apply the Multivariable Fusion Lemma. For every $\langle \xi, \vec{\nu}\rangle = \langle \xi, \nu_0, \dots, \nu_k\rangle$, let–

$$e(\xi,\vec{\nu}) = \{r \in P \setminus (\nu_k+1) : \exists \alpha < h(\xi,\vec{\nu}),\ r \Vdash \underset{\sim}{f}(\xi) < \alpha\}$$

Since $h(\xi,\vec{\nu})$ is regular and non-measurable, and by corollary 2.2.7, $e(\xi,\vec{\nu})$ is $\le^*$-dense open above conditions which force that $\vec{\mu}(\xi) = \vec{\nu}$. Indeed, fix a condition $q \in P \setminus (\nu_k+1)$ above p. Denote $h = h(\xi,\vec{\nu})$. First direct extend $q^* \setminus h \ge^* q \setminus h$ such that–

$$q \restriction_h \Vdash \alpha < h,\ q^* \setminus h \Vdash \underset{\sim}{f}(\xi) = \alpha$$

this is possible because the direct extension order of $P \setminus h$ is more than h-closed, since h is non-measurable. Now, apply corollary 2.2.7 to direct extend $q^* \restriction_h \ge^* p \restriction_h$, such that, for some $\alpha < h$, $q^* \Vdash \underset{\sim}{f}(\xi) < \alpha$.

Let $p^* \ge^* p$ and T be a C-tree, such that, for every $\langle \xi, \nu_0, \dots, \nu_k\rangle \in T$ which is admissible for p^*,

$$(p^{*\frown}\langle \xi,\vec{\nu}\rangle) \restriction_{\nu_k+1} \Vdash \exists \alpha < h(\xi,\vec{\nu}),\ (p^{*\frown}\langle \xi,\vec{\nu}\rangle) \setminus (\nu_k+1) \Vdash \underset{\sim}{f}(\xi) < \alpha$$

We can assume that $p^* \in G$, by applying the same argument above any condition which extends p. For every $\langle \xi,\vec{\nu}\rangle \in T$, pick a P_{ν_k+1}-name $\underset{\sim}{\alpha}(\xi,\vec{\nu})$ for the above α.

Given $\langle \xi, \nu_0, \dots, \nu_k\rangle \in T$ which is admissible for p^*, let–

$$\delta(\xi,\vec{\nu}) = \sup\{\gamma < h(\xi,\vec{\nu}) : \exists r \ge p^{*\frown}\langle \xi,\vec{\nu}\rangle \restriction_{\nu_k+1},\ r \Vdash \underset{\sim}{\alpha}(\xi,\vec{\nu}) = \gamma\}$$

Note that $\delta(\xi,\vec{\nu}) < h(\xi,\vec{\nu})$ since the forcing $P\restriction_{\nu_k+1}$ has cardinality strictly below $h(\xi,\vec{\nu})$ (we can assume that $h(\xi,\vec{\nu}) > |\nu_k|^+$ since $\lambda > \mu^+_{\alpha_k}$. The latter can be easily verified since k_α maps μ_{α_k}, and its successor, to themselves, and $\lambda = \mathrm{crit}(k_\alpha)$). It follows that for every $\langle\xi,\vec{\nu}\rangle \in T$ which is admissible for p^*,

$$p^{*\frown}\langle\xi,\vec{\nu}\rangle \Vdash \underset{\sim}{f}(\xi) < \delta(\xi,\vec{\nu})$$

and the mapping $\langle\xi,\vec{\nu}\rangle \mapsto \delta(\xi,\vec{\nu})$ lies in V.

Apply remark 2.4.13, and let us assume that–

$$\{\xi<\kappa\colon \langle\xi,\vec{\mu}(\xi)\rangle \in T \text{ is admissible for } p^* \text{ and } p^{*\frown}\langle\xi,\vec{\mu}(\xi)\rangle \in G\} \in W$$

For every ξ in the above set, $f(\xi) < \delta(\xi,\vec{\mu}(\xi))$ holds in $V[G]$. Thus, in $M[H]$,

$$\lambda = [f]_W < [\xi \mapsto \delta(\xi,\vec{\mu}(\xi))]_W = k_\alpha\left(j_\alpha\left(\langle\xi,\vec{\nu}\rangle \mapsto \delta(\xi,\vec{\nu})\right)(\kappa,\mu_{\alpha_0},\dots,\mu_{\alpha_k})\right)$$

But this is a contradiction since $\lambda = \mathrm{crit}(k_\alpha)$ and–

$$j_\alpha\left(\langle\xi,\vec{\nu}\rangle \mapsto \delta(\xi,\vec{\nu})\right)(\kappa,\mu_{\alpha_0},\dots,\mu_{\alpha_k}) < j_\alpha(h)(\kappa,\vec{\mu}) = \lambda$$

$\square$

Lemma 2.4.15. *Denote $\lambda^* = k_\alpha(\lambda)$. Then λ appears in the Prikry sequence of λ^* in $M[H]$.*

Proof. In $M[H]$, denote by t_λ the finite initial segment of the Prikry sequence of λ^*, which contains all the elements strictly below λ. By modifying the nice sequence $\langle\alpha_0,\dots,\alpha_k\rangle$, we can assume that there exists a function $\langle\xi,\vec{\nu}\rangle \mapsto t_\lambda(\xi,\vec{\nu})$ in V, such that $t_\lambda = j_\alpha\left(\langle\xi,\vec{\nu}\rangle \mapsto t_\lambda(\xi,\vec{\nu})\right)(\kappa,\vec{\mu})$. Assume that t_λ has length $n^* < \omega$.

Define (in $V[G]$) a function $\xi \mapsto \lambda(\xi)$ with domain κ, such that for each $\xi < \kappa$, $\lambda(\xi)$ is the (n^*+1)-th element in the Prikry sequence of $h(\xi,\mu_{\alpha_0}(\xi),\dots,\mu_{\alpha_k}(\xi))$. Clearly $[\xi \mapsto \lambda(\xi)]_W \geq \lambda$, as it is the first element which appears after t_λ in the Prikry sequence of λ^*. Thus, it suffices to prove that for every $\eta < [\xi \mapsto \lambda(\xi)]_W$, $\eta < \lambda$.

Assume that $f \in V[G]$ is a function such that $\eta = [f]_W < [\xi \mapsto \lambda(\xi)]_W$. Assume that for every $\xi < \kappa$, $f(\xi) < \lambda(\xi)$. Let $p \in G$ be a condition which forces this.

Let us apply the Multivariable Fusion Lemma. For every $\langle\xi,\vec{\nu}\rangle = \langle\xi,\nu_0,\dots,\nu_k\rangle$, let–

$$e(\xi,\vec{\nu}) = \{r \in P\setminus(\nu_k+1) : \exists\alpha < h(\xi,\vec{\nu}),\ r \Vdash \text{ if } t_\lambda(\xi,\vec{\nu}) \text{ is an initial segment}$$
$$\text{of the Prikry sequence of } h(\xi,\vec{\nu}),\ \text{then } \underset{\sim}{f}(\xi) < \alpha\}$$

We argue that $e(\xi,\vec{\nu})$ is $\leq^*$ dense above every condition which forces that $\underset{\sim}{\vec{\mu}}(\xi)=\vec{\nu}$. Let $p\in P\setminus(\nu_k+1)$ be such a condition. Denote for simplicity $h=h(\xi,\vec{\nu})$. First, direct extend $p\restriction_{(\nu_k+1,h)}$ such that it decides whether $t_\lambda(\xi,\vec{\nu})$ and t^p_h are compatible:

1. If $h\notin\operatorname{supp}(p)$, direct extend p such that $t^p_h=t_\lambda(\xi,\vec{\nu})$.
2. If $t_\lambda(\xi,\vec{\nu})$ and t^p_h are incompatible, pick $\alpha=0$.
3. If t^p_h is a strict initial segment of $t_\lambda(\xi,\vec{\nu})$, direct extend by replacing $\underset{\sim}{A}^p_h$ with $\underset{\sim}{A}^p_h\setminus\max(t_\lambda(\xi,\vec{\nu}))+1$. Then take $\alpha=0$.
4. If $t_\lambda(\xi,\vec{\nu})$ is strictly an initial segment of t^p_h, direct extend $p^*\restriction_{(\nu_k+1,h)}\geq^* p\restriction_{(\nu_k+1,h)}$ such that for some $\alpha<h$, $p^*\restriction_{(\nu_k+1,h)}$ forces that the (n^*+1)-th element of t^p_h is bounded by α. It will follow that $p^*\restriction_h{}^\frown p\setminus h\Vdash\underset{\sim}{\lambda}(\xi)<\alpha$.

Let us assume that p has already been direct extended as above, and $p\restriction_h\Vdash t^p_h=t_\lambda(\xi,\vec{\nu})$. Direct extend $p^*\setminus(h+1)\geq^* p\setminus(h+1)$ such that–

$$p\restriction_{h+1}\Vdash\exists\delta<h,\ p^*\setminus(h+1)\Vdash\underset{\sim}{f}(\xi)=\delta$$

Since it is forced that $f(\xi)<\lambda(\xi)$, $p\restriction_h$ forces that for every $\alpha\in\underset{\sim}{A}^p_h$ there exists an ordinal $\delta_\alpha<\alpha$ and a set B_α such that–

$$\langle t_\lambda(\xi,\vec{\nu})^\frown\langle\alpha\rangle,B_\alpha\rangle^\frown p^*\setminus(h+1)\Vdash\underset{\sim}{f}(\xi)=\delta_\alpha$$

Thus, there exists a set $B\in\underset{\sim}{U}^*_h$, $B\subseteq\underset{\sim}{A}^p_h\cap(\triangle_{\alpha<h}B_\alpha)$, and an ordinal $\delta<h$, such that for every $\alpha\in B$, $\delta_\alpha=\delta$. Direct extend $p^*(h)\geq^* p(h)$ such that $\underset{\sim}{A}^{p^*}_h=B$. Finally, direct extend $p^*\restriction_h\geq^* p\restriction_h$ such that, for some $\alpha<h$ (in $V^{P_{\nu_k+1}}$), $p^*\restriction_h\Vdash\underset{\sim}{\delta}<\alpha$. Thus, $p^*\geq^* p$ forces that $\underset{\sim}{f}(\xi)<\alpha$.

Now, fix $p^*\in G$ and a C-tree T such that for every $\langle\xi,\vec{\nu}\rangle\in T$,

$$(p^{*\frown}\langle\xi,\vec{\nu}\rangle)\restriction_{\nu_k+1}\Vdash\exists\alpha<h(\xi,\vec{\nu}),\ \text{if } t_\lambda(\xi,\vec{\nu})\ \text{is an initial segment of the Prikry sequence of } h(\xi,\vec{\nu})\ \text{then}\ (p^{*\frown}\langle\xi,\vec{\nu}\rangle)\setminus(\nu_k+1)\Vdash\underset{\sim}{f}(\xi)<\alpha$$

Let $\underset{\sim}{\alpha}(\xi,\vec{\nu})$ be a name for the above α, and define–

$$\delta(\xi,\vec{\nu})=\sup\{\gamma<h(\xi,\vec{\nu}):\exists r\geq p^{*\frown}\langle\xi,\vec{\nu}\rangle\restriction_{\nu_k+1},\ r\Vdash\underset{\sim}{\alpha}(\xi,\vec{\nu})=\gamma\}$$

as before, $\delta(\xi,\vec{\nu})<h(\xi,\vec{\nu})$.

Finally, work in $V[G]$. As before,

$$\{\xi < \kappa \colon \langle \xi, \vec{\mu}(\xi)\rangle \in T \text{ is admissible for } p^* \text{ and } p^{*\frown}\langle \xi, \vec{\mu}(\xi) \in G\rangle\} \in W$$

Moreover,

$$\{\xi < \kappa \colon t_\lambda(\xi, \vec{\mu}(\xi)) \text{ is an initial segment of the Prikry sequence of } h(\xi, \vec{\mu}(\xi))\} \in W$$

Thus, in $M[H]$,

$$[f]_W < [\xi \mapsto \delta(\xi, \vec{\mu}(\xi))]_W = k_\alpha(j_\alpha(\langle \xi, \vec{\nu}\rangle \mapsto \delta(\xi, \vec{\nu}))(\kappa, \vec{\mu}))$$

but $j_\alpha(\langle \xi, \vec{\nu}\rangle \mapsto \delta(\xi, \vec{\nu}))(\kappa, \vec{\mu}) < \lambda$ since $\delta(\xi, \vec{\nu}) < h(\xi, \vec{\nu})$ for every $\xi, \vec{\nu}$. Thus, in $M[H]$, $\eta = [f]_W < \lambda$, as desired.

□

Let us denote $U_\lambda = \{X \subseteq \lambda \colon \lambda \in k_\alpha(X)\} \cap M_\alpha$. This is an M_α-ultrafilter. We will eventually prove that $\lambda = \mu_\alpha$, and then $U_\lambda = U_{\mu_\alpha}$ will be the M_α-ultrafilter which is used to form $M_{\alpha+1}$ in the iterated ultrapower.

Lemma 2.4.16. *$U_\lambda \in M_\alpha$. Moreover, it is a normal measure of Mitchell order 0 there.*

Proof. The proof follows from a pair of claims.

Claim 2.4.17. *There exist $p \in G$ and a set $\mathcal{F} \in M_\alpha$ of normal measures on λ, each of Mitchell order 0, such that $|\mathcal{F}| < \lambda$ and $j_\alpha(p)^\frown\langle \kappa, \vec{\mu}\rangle \restriction_\lambda \Vdash j_\alpha(\underset{\sim}{\mathcal{U}})(\lambda) \in \mathcal{F}$.*

Proof. In V, for every measurable $x < \kappa$, let S_x be an enumeration of all the normal measures on x of order 0.

We claim that there exists $p \in G$ and a C-tree T, such that for every $\langle \xi, \vec{\nu}\rangle \in T$ which is admissible for p, there exists a set of ordinals $A(\xi, \vec{\nu})$ with $|A(\xi, \vec{\nu})| < h(\xi, \vec{\nu})$, such that–

$$p^\frown\langle \xi, \vec{\nu}\rangle \restriction_{h(\xi,\vec{\nu})} \Vdash \underset{\sim}{U}_{h(\xi,\vec{\nu})} \in \left(S_{h(\xi,\vec{\nu})}\right)'' A(\xi, \vec{\nu})$$

This follows from the Multivariable Fusion Lemma. Fix $\langle \xi, \vec{\nu}\rangle = \langle \xi, \nu_0, \ldots, \nu_k\rangle$ and denote for simplicity $h = h(\xi, \vec{\nu})$. Consider–

$$\begin{aligned} e(\xi, \vec{\nu}) = \{r \in P \setminus \nu_k + 1 \colon & \text{ there exists a set of ordinals } A \text{ with } |A| < h \\ & \text{such that } r \restriction_h \Vdash \underset{\sim}{U}_h \in S_h{}'' A\} \end{aligned}$$

Let us argue that $e(\xi,\vec{\nu})$ is $\leq^*$-dense open above conditions which force that $\underset{\sim}{\mu}(\xi)=\vec{\nu}$. Let p be such a condition. Note that every condition in P_h, and $p\restriction_h$ in particular, forces that there exists an ordinal α such that $\underset{\sim}{U}_h = S_h(\alpha)$; Now, direct extend $p^*\restriction_h \geq^* p\restriction_h$ such that for some A of cardinality less then h, $p^*\restriction_h \Vdash \underset{\sim}{\alpha}\in A$.

Now pick $p\in G$ and a C-tree T as above. Then for every $\langle\xi,\vec{\nu}\rangle\in T$ which is admissible for p,

$$\begin{aligned}&p^\frown\langle\xi,\vec{\nu}\rangle\restriction_{\nu_k+1}\Vdash \text{ there exists a set of ordinals } A \text{ with } |A|<h(\xi,\vec{\nu}),\\&\text{such that } p^\frown\langle\xi,\vec{\nu}\rangle\restriction_{(\nu_k,h(\xi,\vec{\nu}))}\Vdash \underset{\sim}{U}_{h(\xi,\vec{\nu})}\in S_{h(\xi,\vec{\nu})}{}''A\end{aligned}$$

For every such $\langle\xi,\vec{\nu}\rangle\in T$, let $\underset{\sim}{A}(\xi,\vec{\nu})$ be a P_{ν_k+1}-name for A above, and let–

$$A^*(\xi,\vec{\nu})=\{\gamma\colon \exists r\geq p^\frown\langle\xi,\vec{\nu}\rangle,\ r\Vdash \gamma\in\underset{\sim}{A}(\xi,\vec{\nu})\}$$

Then $|A^*(\xi,\vec{\nu})|<h(\xi,\vec{\nu})$, and–

$$p^\frown\langle\xi,\vec{\nu}\rangle\restriction_{h(\xi,\vec{\nu})}\Vdash \underset{\sim}{U}_{h(\xi,\underset{\sim}{\vec{\mu}}(\xi))}\in S_{h(\xi,\underset{\sim}{\vec{\mu}}(\xi))}{}''A^*(\xi,\underset{\sim}{\vec{\mu}}(\xi))$$

Let $A^* = j_\alpha(\langle\xi,\vec{\nu}\rangle\mapsto A^*(\xi,\vec{\nu}))$. Denote $\mathcal{F}=((j_\alpha(S))_\lambda)''A^*$. Then $|A^*|<\lambda$ and thus $|\mathcal{F}|<\lambda$. $j_W(p)^\frown\langle\kappa,\vec{\mu}\rangle\restriction_{k_\alpha(\lambda)}$ forces that $j_W(\underset{\sim}{\mathcal{U}})(k_\alpha(\lambda))\in k_\alpha(\mathcal{F})$. Thus, by elementarity of k_α, $j_\alpha(p)^\frown\langle\kappa,\vec{\mu}\rangle\restriction_\lambda$ forces that $j_\alpha(\underset{\sim}{\mathcal{U}})(\lambda)\in\mathcal{F}$. □

Claim 2.4.18. *Assume that $B\in U_\lambda$. Then there exists $p\in G$ such that $j_\alpha(p)^\frown\langle\kappa,\vec{\mu}\rangle\restriction_\lambda\Vdash B\in j_\alpha(\underset{\sim}{\mathcal{U}})(\lambda)$.*

Proof. Let $\langle\xi,\vec{\nu}\rangle\mapsto B(\xi,\vec{\nu})$ be a function in V such that–

$$B=j_\alpha(\langle\xi,\vec{\nu}\rangle\mapsto B(\xi,\vec{\nu}))(\kappa,\vec{\mu})$$

(we assumed, without loss of generality, that B can be represented using $\vec{\mu}$; else, change $\vec{\mu}$). Let $n^*<\omega$ be the coordinate in which λ appears in the Prikry sequence of λ^*. In $V[G]$, denote by $\lambda(\xi)$ the n^*-th element in the Prikry sequence of $h(\xi,\vec{\mu}(\xi))$, so that $[\xi\mapsto\lambda(\xi)]_W=\lambda$.

As usual, we apply the Multivariable Fusion Lemma. Given $\langle\xi,\vec{\nu}\rangle$, let–

$$\begin{aligned}e(\xi,\vec{\nu})=\{r\in P\setminus\nu_k+1\colon\ & r\restriction_{h(\xi,\vec{\nu})} \text{ decides whether } B(\xi,\vec{\nu})\in\underset{\sim}{U}_{h(\xi,\vec{\nu})}, \text{ and there}\\&\text{exists a bounded subset } A\subseteq h(\xi,\vec{\nu}) \text{ such that the following holds:}\\&\text{If } r\restriction_{h(\xi,\vec{\nu})}\Vdash B(\xi,\vec{\nu})\in\underset{\sim}{U}_{h(\xi,\vec{\nu})},\ \underset{\sim}{\lambda}(\xi)\in A\cup B(\xi,\vec{\nu});\ \text{else,}\\&\underset{\sim}{\lambda}(\xi)\in A\cup(h(\xi,\vec{\nu})\setminus B(\xi,\vec{\nu}))\}\end{aligned}$$

$e(\xi,\vec{\nu})$ is $\leq^*$ dense open above any condition which forces that $\underset{\sim}{\vec{\mu}}(\xi)=\vec{\nu}$. Indeed, let $p\in P\setminus\nu_k+1$ be such a condition. Denote $h=h(\xi,\vec{\nu})$. Direct extend $p^*\restriction_h\geq^* p\restriction_h$ such that it decides the length of t^p_h and which of the sets $B(\xi,\vec{\nu})$, $h\setminus B(\xi,\vec{\nu})$ belongs to $\underset{\sim}{U}_h$:

1. If the length of t^p_h is $\geq n^*$, direct extend $p^*\restriction_h\geq^* p\restriction_h$ such that for some bounded subset $A\subseteq\lambda$, $p^*\restriction_h$ forces that the n^*-th element in the Prikry sequence of h belongs to A.
2. Otherwise, $t^p_h=t_\lambda(\xi,\vec{\nu})$. In this case, direct extend and shrink $\underset{\sim}{A}^p_h$ such that it is entirely contained in exactly one of the sets $B(\xi,\vec{\nu})$, $h\setminus B(\xi,\vec{\nu})$.

The condition p^* obtained this way is as desired.

Now pick $p\in G$ and a C-tree T such that for every $\langle\xi,\vec{\nu}\rangle\in T$ which is admissible for p,

$$\begin{aligned}&p^\frown\langle\xi,\vec{\nu}\rangle\restriction_{\nu_k+1}\Vdash p^\frown\langle\xi,\vec{\nu}\rangle\restriction_{(\nu_k,h(\xi,\vec{\nu}))}\text{ decides whether } B(\xi,\vec{\nu})\in\underset{\sim}{U}_{h(\xi,\vec{\nu})},\\&\text{and there exists a bounded subset } A\subseteq h(\xi,\vec{\nu})\text{ such that}\\&p^\frown\langle\xi,\vec{\nu}\rangle\restriction_{(\nu_k,h(\xi,\vec{\nu}))}\Vdash\underset{\sim}{\lambda}(\xi)\text{ belongs to exactly one of the sets } A\cup B(\xi,\vec{\nu})\\&\text{or } A\cup(h(\xi,\vec{\nu})\setminus B(\xi,\vec{\nu}))\,,\text{ according to the above decision.}\end{aligned}$$

For every such $\langle\xi,\vec{\nu}\rangle\in T$, let $\underset{\sim}{A}(\xi,\vec{\nu})$ be a P_{ν_k+1}-name for A above, and let–

$$A^*(\xi,\vec{\nu})=\{\gamma\colon\exists r\geq p^\frown\langle\xi,\vec{\nu}\rangle,\ r\Vdash\gamma\in\underset{\sim}{A}(\xi,\vec{\nu})\}$$

Then $A^*(\xi,\vec{\nu})$ is a bounded subset of $h(\xi,\vec{\nu})$.

We argue that $j_\alpha(p)^\frown\langle\kappa,\vec{\mu}\rangle\Vdash B\in j_\alpha(\underset{\sim}{\mathcal{U}})(\lambda)$. Work in $V[G]$. Then for a set of ξ-s in W, $\langle\xi,\vec{\mu}(\xi)\rangle\in T$ is admissible for p. Thus,

$$p^\frown\langle\xi,\vec{\mu}(\xi)\rangle\restriction_{h(\xi,\vec{\mu}(\xi))}\parallel B(\xi,\vec{\mu}(\xi))\in\underset{\sim}{U}(h(\xi,\vec{\mu}(\xi)))$$

We argue that for a set of ξ-s in W,

$$p^\frown\langle\xi,\vec{\mu}(\xi)\rangle\restriction_{h(\xi,\vec{\mu}(\xi))}\Vdash B(\xi,\vec{\mu}(\xi))\in\underset{\sim}{U}(h(\xi,\vec{\mu}(\xi)))$$

Assume otherwise. Then–

$$\{\xi<\kappa\colon\lambda(\xi)\in A(\xi,\vec{\mu}(\xi))\cup(h(\xi,\vec{\nu})\setminus B(\xi,\vec{\mu}(\xi)))\}\in W$$

However, this cannot hold:

1. If $\{\xi < \kappa \colon \lambda(\xi) \in A(\xi, \vec{\mu}(\xi))\} \in W$, then, since $|A(\xi, \vec{\nu})| < h(\xi, \vec{\nu})$ for every $\xi, \vec{\nu}$, it follows that $\lambda \in \mathrm{Im}(k_\alpha)$, which is a contradiction.

2. Else, $\{\xi < \kappa \colon \lambda(\xi) \in h(\xi, \vec{\mu}(\xi) \setminus B(\xi, \vec{\mu}(\xi)))\}$. But then $\lambda \notin k_\alpha(B)$, contradicting the fact that $B \in U_\lambda$.

Thus, $j_W(p)^\frown \langle \kappa, \vec{\mu} \rangle \restriction_{k_\alpha(\lambda)} \Vdash k_\alpha(B) \in j_W(\underset{\sim}{\mathcal{U}})(k_\alpha(\lambda))$ and by elementarity of k_α, $j_\alpha(p)^\frown \langle \kappa, \vec{\mu} \rangle \restriction_\lambda \Vdash B \in j_\alpha(\underset{\sim}{\mathcal{U}})(\lambda)$, as desired. $\square$

Fix now a set $\mathcal{F}$ and a condition $p \in G$ as in the first claim. Since $\mathcal{F}$ is a sequence of normal measures on λ of cardinality $< \lambda$, there exists a partition $\langle B_F \colon F \in \mathcal{F} \rangle$ of λ such that for every $F \in \mathcal{F}$, $B_F \in F$. $|\mathcal{F}| < \lambda$, and thus there exists a unique $F^* \in \mathcal{F}$ such that $\lambda \in k_\alpha(B_{F^*})$. We denote for simplicity $B^* = B_{F^*}$.

By second claim, applied for the set $B^* \in U_\lambda$, there exists $p^* \in G$ above p such that $j_\alpha(p^*)^\frown \langle \kappa, \vec{\mu} \rangle \Vdash j_\alpha(\underset{\sim}{\mathcal{U}})(\lambda) = F^*$.

Finally, $F^* = U_\lambda$ follows. Indeed, let $X \in F^*$. By the second claim, for every $X \in U_\lambda$, there exists $p \in G$ such that $j_\alpha(p)^\frown \langle \kappa, \vec{\mu} \rangle \Vdash X \in j_\alpha(\underset{\sim}{\mathcal{U}})(\lambda)$. Without loss of generality, p extends p^* which was chosen in the previous paragraph, and thus $j_\alpha(p) \Vdash X \in F^*$. Since X and F^* are elements of M_α (and not names), it follows that $X \in F^*$. $\square$

Corollary 2.4.19. *In $M[H]$, $j_W(\mathcal{U})(k_\alpha(\lambda)) = k_\alpha(U_\lambda)$. In particular, if $\mathcal{U} \in V$, then $j_\alpha(\mathcal{U})(\lambda) = U_\lambda$.*

Proof. This follows since, by the proof of the previous lemma, there exists $p \in G$ such that $j_\alpha(p)^\frown \langle \kappa, \vec{\mu} \rangle \Vdash j_\alpha(\underset{\sim}{\mathcal{U}})(\lambda) = U_\lambda$. Now apply $k_\alpha \colon M_\alpha \to M$ and use the fact that $j_W(p)^\frown \langle \kappa, \vec{\mu} \rangle \in H$. $\square$

Lemma 2.4.20. *In V, $cf(\lambda) \geq \kappa^+$.*

Proof. Denote $M' = \mathrm{Ult}(M_\alpha, U_\lambda)$, and let $j' \colon V \to M'$, be defined as follows:

$$j' = j^{M_\alpha}_{U_\lambda} \circ j_\alpha$$

There exists an elementary embedding $k' \colon M' \to M$, defined as follows:

$$k'(j'(f)(\kappa, \mu_{i_0}, \ldots, \mu_{i_m}, \lambda)) = j_W(f)(\kappa, \mu_{i_0}, \ldots, \mu_{i_m}, \lambda)$$

for every $f \in V$ and $i_0 < \ldots < i_m < \alpha$.

Since U_λ was derived from k_α, $k' \colon M' \to M$ is elementary (the proof is the same as in lemma 2.4.3). It's not hard to verify that $\text{crit}\,(k') > \lambda$. Therefore, λ, which is a non-measurable inaccessible cardinal in M', is still a non-measurable inaccessible cardinal in M.

Let us argue that λ is regular in $M[H]$. Split $H = H_\lambda * H'$, where $H_\lambda \subseteq j_W(P)\restriction_\lambda$. If λ changes its cofinality in $M[H]$, then it changes its cofinality in $M[H_\lambda]$ (since the upper forcing has a direct extension order which is more than λ–closed). However, by corollary 2.2.8, λ is regular in $M[H_\lambda]$.

It follows that, in $V[G]$, $\text{cf}(\lambda) \geq \kappa^+$. Thus, in V, $\text{cf}\,(\lambda) \geq \kappa^+$. □

Corollary 2.4.21. *$crit\,(k_\alpha) = \mu_\alpha$.*

Proof. It suffices to prove that $\text{crit}\,(k_\alpha) \geq \mu_\alpha$. Denote $\bar{\mu} = \sup\{\mu_\beta \colon \beta < \alpha\}$. We already argued that $\text{crit}(k_\alpha) \geq \bar{\mu}$.

By all the properties proved so far, $\text{crit}\,(k_\alpha)$ is a measurable cardinal in M_α, with cofinality $> \kappa$ in V. By the definition, $\mu_\alpha \geq \bar{\mu}$ is the least such cardinal. Thus, $\text{crit}\,(k_\alpha) \geq \mu_\alpha$. □

This finishes the inductive proof of properties (A)-(E). We are now prepared to finish the proof of Theorem 2.4.1:

Proof of theorem 2.4.1. Recall that $\kappa^* = j_U(\kappa)$. It's not hard to prove by induction that, for every $\alpha < \kappa^*$, $\mu_\alpha < \kappa^*$. Note that $j_{\kappa^*}(\kappa) = \kappa^*$, since j_{κ^*} is an iterated ultrapower with measures on measurables below κ^*. Since κ^* is measurable in each step, it does not move in $j_{1,\kappa^*} \colon M_U \to M_{\kappa^*}$.

Recall the embedding $k_{\kappa^*} \colon M_{\kappa^*} \to M$, defined as follows:

$$k_{\kappa^*}\left(j_{\kappa^*}(f)\left(\kappa, \mu_{i_0}, \ldots, \mu_{i_m}\right)\right) = j_W\left(f\right)\left(\kappa, \mu_{i_0}, \ldots, \mu_{i_m}\right)$$

for every $f \in V$, $m < \omega$ and $i_0, \ldots, i_m < \kappa^*$. As in lemma 2.4.3, k_{κ^*} is elementary, $k_{\kappa^*} \circ j_{\kappa^*} = j_W \restriction_V$ and $\text{crit}\,(k_{\kappa^*}) \geq \kappa^*$.

In order to prove that $M = M_{\kappa^*}$, $j_{\kappa^*} = j_W \restriction_V$ and $k^* = j_W(\kappa)$, it suffices to prove that $k_{\kappa^*} \colon M_{\kappa^*} \to M$ is the identity. Thus, it suffices to prove that for every ordinal η, $\eta \in \text{Im}\,(k_{\kappa^*})$. Assume that $g \in V[G]$ is a function such that $\eta = [g]_W$. Let $p \in G$ be a condition. By lemma 2.2.6, there exists a condition $p \leq p^* \in G$, a function $\xi \mapsto A_\xi$ in V and a club $C \subseteq \kappa$ such that, for every $\xi \in C$, $|A_\xi| < \kappa$, and $p^* \Vdash \underset{\sim}{g}(\xi) \in A_\xi$. Then $j_W\left(p^*\right) \in H$ forces that $\eta = [\xi \mapsto \underset{\sim}{g}(\xi)]_W \in [\xi \mapsto A_\xi]_W = k_{\kappa^*}\left(j_{\kappa^*}\left(\xi \mapsto A_\xi\right)(\kappa)\right)$; but $|j_{\kappa^*}\left(\xi \mapsto A_\xi\right)(\kappa)| < j_{\kappa^*}(\kappa) = \kappa^* \leq \text{crit}\,(k_{\kappa^*})$. Therefore, $\eta \in \text{Im}\,(k_{\kappa^*})$ as desired.

Finally, note that if $\mathcal{U} \in V$, then by corollary 2.4.19, $U_{\mu_\alpha} = j_\alpha(\mathcal{U})(\mu_\alpha) \in M_\alpha$ for every $\alpha < \kappa$, and thus the iteration j_{κ^*} is definable over V. Also, $M = M_{\kappa^*}$ is a class of M. $\square$

We finish this section with several remarks about definability of $j_W \restriction_V$ in V.

First, $\mathcal{U} \in V$ holds under the assumption that the Mitchell order is linear (or there exists a unique normal measure of Mitchell order 0 an any measurable cardinal): this holds since $\mathcal{U}$, in this case, is the sequence of Measures of order 0. Thus, in this case, $j_W \restriction_V$ is a definable class of V. This proves corollary 2.1.3.

The condition $\mathcal{U} \in V$ is sufficient but not necessary for the definability of $j_W \restriction_V$. For instance, let $\eta \in \Delta$ be the first measurable. Assume that, in V, there are infinitely many measurables which carry η measures of Mitchell order 0. Take in V an enumeration $\langle \alpha_n : n < \omega \rangle$ of the first ω such measurables above η. For every $n < \omega$, let $\langle F^\xi_{\alpha_n} : \xi < \eta \rangle$ be an enumeration of η-many measures of Mitchell order 0 on α_n. Let P be the forcing notion which uses, at stage α_n, the unique normal measure which extends $F^{\eta_n}_{\alpha_n}$, where $\eta_n < \eta$ is the n-th element in the Prikry sequence of η. For every other measurable α, use a measure which extends the least measure on α of Mitchell order 0 with respect to a prescribed well order of V_κ. So $\mathcal{U} \notin V$, since it codes the Prikry sequence of η. However, $j_W \restriction_V$ is definable in V, by repeating the argument of corollary 2.4.19, replacing $\mathcal{U}$ with $\mathcal{U} \restriction_{\Delta \setminus \{\alpha_n : n < \omega\}} = \langle U_\alpha : \alpha \in \Delta \setminus \{\alpha_n : n < \omega\} \rangle \in V$. More generally, the following holds, and is proved similarly to corollary 2.4.19:

Lemma 2.4.22. *Assume that for some $\xi < \kappa$, $\mathcal{U} \setminus \xi = \langle U_\alpha : \alpha \in \Delta \setminus \xi \rangle \in V$. Then $j_W \restriction_V$ is definable in V.*

Remark 2.4.23. *Let $A \subseteq \Delta$ be a set such that, for every $\alpha < \kappa^*$, $\mu_\alpha \in j_\alpha(A)$. If $\mathcal{U} \restriction_A = \langle U_\alpha : \alpha \in A \rangle \in V$, then $j_W \restriction_V$ is definable in V, and again, this is proved by repeating the argument of corollary 2.4.19, replacing $\mathcal{U}$ with $\mathcal{U} \restriction_A$. This seems like an improvement of the previous lemma; however, we will prove in lemma 2.5.12, that a set A satisfies that $\mu_\alpha \in j_\alpha(A)$ for every $\alpha < \kappa^*$, if and only if, for some $\xi < \kappa$, $\Delta \setminus \xi \subseteq A$.*

By lemma 2.4.22, definability of $j_W \restriction_V$ in V follows from the assumption that $j_W(\mathcal{U}) \setminus \kappa \in M$. In the next section we will prove that the other direction is not necessarily true.

2.5 A General Analysis Of Iterated Ultrapowers

Our main goal in this section is to simplify the presentation of $j_W \restriction_V$ provided in the previous section; for instance, we will provide a simpler characterization of the critical points μ_α. Simultaneously, we describe in detail how the Prikry sequences, added to measurables of M above κ, look like: up to a finite initial segment, those are sequences of critical points of an iterated ultrapower, generated over some finite sub-iteration of $\langle M_\alpha \colon \alpha < \kappa^* \rangle$, using a single measure. It will follow that every Prikry sequence, added in $M[H]$ for a measurable cardinal above κ, already belongs to V.

Our goals are lemma 2.5.6 and corollaries 2.5.7, 2.5.9 and 2.5.13.

We start by studying linear iterations of V in more general settings. Let us assume that κ^* is an ordinal, and $\langle M_\alpha \colon \alpha < \kappa^* \rangle$ is a linear iteration of V, by normal measures of Mitchell order 0. More specifically, we assume that $M_0 = \mathrm{Ult}(V, U)$ where U is a measure of Mitchell order 0 on some measurable κ; in successor steps, $M_{\alpha+1} = \mathrm{Ult}(M_\alpha, U_{\mu_\alpha})$, where $U_{\mu_\alpha} \in M_\alpha$ is a normal measure of order 0 on some measurable μ_α; at limit steps a direct limit is taken. We assume also that the iteration is normal in the sense that $\langle \mu_\alpha \colon \alpha < \kappa^* \rangle$ is increasing. We do not assume that the entire iteration is definable in V. Finally, we denote $M = M_{\kappa^*}$.

We begin by observing that every finite nice sequence corresponds to a finite iteration of V which naturally embeds in M. Assume that $\langle \alpha_0, \ldots, \alpha_m \rangle$ is a nice sequence below some ordinal $\alpha < \kappa^*$. Recall that this means that, every $0 \le k \le m$, there are functions $g_k, F_k \in V$ such that–

$$\mu_{\alpha_k} = j_{\alpha_k}(g_k)\left(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_{k-1}}\right)$$

$$U_{\mu_{\alpha_k}} = j_{\alpha_k}(F_k)\left(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_{k-1}}\right)$$

(for $k = 0$, $\mu_{\alpha_0} = j_{\alpha_0}(g_0)(\kappa)$ and $U_{\mu_{\alpha_0}} = j_{\alpha_0}(F_0)(\kappa)$).

We define a finite iteration $\langle N_k \colon k \le m+1 \rangle$ of V, for each $k \le m$ an embedding $i_k \colon V \to N_k$, a cardinal λ_k measurable in N_k and a measure $W_k \in N_k$ on it of order 0.

First, let $N_0 \simeq \mathrm{Ult}(V, U)$, $i_0 \colon V \to M_0$ the ultrapower embedding, $\lambda_0 = i_0(g_0)(\kappa)$ and $W_0 = i_0(F_0)(\kappa)$.

Assume that $k \le m$ and N_k, W_k and λ_k have been defined. Let $N_{k+1} \simeq \mathrm{Ult}(N_k, W_k)$, $i_{k+1} \colon V \to N_{k+1}$, $i_{k+1} = j^{N_k}_{W_k} \circ i_k$, $\lambda_{k+1} = i_{k+1}(g_{k+1})(\kappa, \lambda_0, \ldots, \lambda_k)$ and $W_{k+1} = i_{k+1}(F_{k+1})(\kappa, \lambda_0, \ldots, \lambda_k)$.

Lemma 2.5.1. *Fix a nice sequence $\langle \alpha_0, \ldots, \alpha_m \rangle$ below some $\alpha < \kappa^*$. In the above notations, define*

$k_{m+1}\colon N_{m+1} \to M_\alpha$ *as follows:*

$$k_{m+1}\left(i_{m+1}\left(f\right)\left(\kappa, \lambda_0, \ldots, \lambda_m\right)\right) = j_\alpha(f)\left(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_m}\right)$$

for every $f \in V$. *Then* $k_{m+1}\colon N_{m+1} \to M_\alpha$ *is an elementary embedding, and–*

$$k_{m+1} = \left(j_{\alpha_m+1,\alpha} \circ \ldots \circ j_{\alpha_0+1,\alpha_1} \circ j_{0,\alpha_0}\right) \restriction_{N_{m+1}}$$

Remark 2.5.2. *The iteration* $j_{\alpha_m+1,\alpha} \circ \ldots \circ j_{\alpha_0+1,\alpha_1} \circ j_{0,\alpha_0}$ *above is not necessarily internal to* N_{m+1}*; this means that the sub-iterations* $j_{\alpha_i+1,\alpha_{i+1}}$ *participating in it are iterated ultrapowers as defined over* M_{α_i+1}. *In the proof of the lemma we will show that the external iteration* $j_{\alpha_m+1,\alpha} \circ \ldots \circ j_{\alpha_0+1,\alpha_1} \circ j_{1,\alpha_0}$ *is well defined over* N_{m+1}*, in the sense that for every* $x \in N_{m+1}$*,* $\left(j_{\alpha_{i-1}+1,\alpha_i} \circ \ldots \circ j_{\alpha_0+1,\alpha_1} \circ j_{0,\alpha_0}\right)(x)$ *belongs to* M_{α_i+1}. *Later in this section, we will prove that such an iteration might be an internal iteration of* N_{m+1}*, provided that the initial nice sequence is chosen more carefully.*

Proof. We proceed by induction on m. The induction basis is given for "$m = -1$", namely, the case where the given nice sequence below α is empty. In this case, $N_0 = \mathrm{Ult}\left(V, U\right)$, $i_0 = j_U\colon V \to N_0$ and $k_0\left(i_0(f)(\kappa)\right) = j_\alpha(f)(\kappa)$, and clearly $k_0 = j_{0,\alpha}\colon M_0 \to M_\alpha$.

Assume now that $m < \omega$ and $k_{m+1}\colon N_{m+1} \to M_{\alpha_{m+1}}$ has been constructed (here, the embedding k_{m+1} corresponds to the nice sequence $\langle \alpha_0, \ldots, \alpha_m\rangle$ below α_{m+1}. Namely, by its definition, it satisfies $j_{\alpha_{m+1}} = k_{m+1} \circ i_{m+1}$). Let us argue that $k_{m+2} = j_{\alpha_{m+1}+1,\alpha} \circ k_{m+1} \restriction_{N_{m+2}}$. Indeed, given an arbitrary element $i_{m+2}(f)\left(\kappa, \lambda_0, \ldots, \lambda_m, \lambda_{m+1}\right)$ of N_{m+2},

$$\begin{aligned}
&k_{m+1}\left(i_{m+2}(f)\left(\kappa, \lambda_0, \ldots, \lambda_m, \lambda_{m+1}\right)\right)\\
&= k_{m+1}\left(j^{N_{m+1}}_{W_{m+1}}\left(i_{m+1}(f)\right)\left(\kappa, \lambda_0, \ldots, \lambda_{m+1}\right)\right)\\
&= j_{U_{\mu_{\alpha_{m+1}}}}\left(j_{\alpha_{m+1}}(f)\right)\left(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_m}, \mu_{\alpha_{m+1}}\right)\\
&= j_{\alpha_{m+1}+1}(f)\left(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_m}, \mu_{\alpha_{m+1}}\right)
\end{aligned}$$

In the equality from line 2 to line 3 in the above equation, we used the fact that $W_{m+1} = i_{m+1}\left(F_{m+1}\right)\left(\kappa, \lambda_0, \ldots, \lambda_m\right)$ and $\lambda_{m+1} = i_{m+1}\left(g_{m+1}\right)\left(\kappa, \lambda_0, \ldots, \lambda_m\right)$ for the computation on their values under k_{m+1}. We also used the fact that $j_{\alpha_{m+1}} = k_{m+1} \circ i_{m+1}$ (which follows from the definition of k_{m+1} provided above). Finally, apply $j_{\alpha_{m+1}+1,\alpha}$ on both sides. □

If the sequence $\langle \alpha_0, \ldots, \alpha_m\rangle$ below α is clear from the context, we denote $N^* = N_{m+1}$, $i^* = i_{m+1}\colon V \to N^*$ and $k^* = k_{m+1}\colon N^* \to M_\alpha$. Note that k^* is not necessarily an internal iteration of N^*. Indeed, assume that $\lambda_0 < \mu_{\alpha_0}$ (this happens, e.g., if $\alpha_0 = 1$. In this case, $\lambda_0 = \mu_0$ and $\mu_1 = j_{U_{\mu_0}}(\mu_0) > \mu_0$). If k^* was an internal iteration of N^*, then λ_0 would have to be one of the critical points participating in the iteration, since λ_0 is inaccessible in N^* and $k^*(\lambda_0) = \mu_{\alpha_0}$. However, this is not possible because λ_0 is not measurable in N^*.

Our goal is lemma 2.5.6. In the proof, it will be useful to consider a nice sequence $\langle \alpha_0, \ldots, \alpha_m \rangle$ below α and its associated iteration N^*, such that the embedding $k^*\colon N^* \to M_\alpha$ is an internal iteration of N^*. This will require a more sophisticated choice of the initial nice sequence. The example from the last paragraph offers a lead: we would like $\lambda_k = \mu_{\alpha_k}$ to hold for every $0 \le k \le m$.

Lemma 2.5.3. *k^* is an internal iteration of N^* if and only if, for every $0 \le k \le m$, $\lambda_k = \mu_{\alpha_k}$.*

Proof. Let us assume first that k^* is an iteration of N^*. Then λ_k is a non-measurable inaccessible in N^*, and thus λ_k cannot move by k^*. So $\mu_{\alpha_k} = k^*(\lambda_k) = \lambda_k$.

Let us concentrate on the other direction. Assume that $\lambda_k = \mu_{\alpha_k}$ for every $0 \le k \le m$. $\lambda_0 = \mu_{\alpha_0}$ is measurable in $M_U = N_0$, and thus $j_{0,\alpha_0}(\mu_{\alpha_0}) = \mu_{\alpha_0}$. Also, $j_{0,\alpha_0}(W_0) = U_{\mu_{\alpha_0}}$. Note that–

$$j_\alpha = j_{\alpha_0+1,\alpha} \circ j_{U_{\mu_{\alpha_0}}} \circ j_{\alpha_0} = j_{\alpha_0+1,\alpha} \circ j^{N_1}_{0,\alpha_0} \circ j_{W_0} \circ j_U$$

where $j^{N_1}_{0,\alpha_0}$ is the iterated ultrapower consisting of the same measures as j_{0,α_0}, but acting on N_1. $j^{N_1}_{0,\alpha_0}\colon N_1 \to M_{\alpha_0+1}$ is internal to N_1, and so is $j_{\alpha_0+1,\alpha} \circ j^{N_1}_{0,\alpha_0}$.

We proceed now by induction on m. Assume that $k_{m+1}\colon N_{m+1} \to M_{\alpha_{m+1}}$ is an internal iteration of N_{m+1} (with respect to the nice sequence $\langle \alpha_0, \ldots, \alpha_m \rangle$ below α_{m+1}). Then–

$$U_{\mu_{\alpha_{m+1}}} = k_{m+1}(W_{m+1})$$

and thus–

$$\begin{aligned} j_{\alpha_{m+1}+1} &= j_{U_{\mu_{\alpha_{m+1}}}} \circ k_{m+1} \circ i_{m+1} \\ &= k_{m+1} \restriction_{N_{m+2}} \circ i_{m+2} \\ &= \left(j_{\alpha_m+1,\alpha_{m+1}} \circ \ldots \circ j_{\alpha_0+1,\alpha_1} \circ j_{0,\alpha_0} \right) \restriction_{N_{m+2}} \circ i_{m+2} \end{aligned}$$

Where $\left(j_{\alpha_m+1,\alpha_{m+1}} \circ \ldots \circ j_{\alpha_0+1,\alpha_1} \circ j_{0,\alpha_0} \right) \restriction_{N_{m+2}}$ above is an internal iteration of N_{m+2}, since W_{m+1} is a measure over $\lambda_{m+1} = \mu_{\alpha_{m+1}}$, and lies strictly above all the participating critical points. Thus, the embedding k_{m+2}, obtained by applying $j_{\alpha_{m+1}+1,\alpha}$ on $\left(j_{\alpha_m+1,\alpha_{m+1}} \circ \ldots \circ j_{\alpha_0+1,\alpha_1} \circ j_{0,\alpha_0} \right) \restriction_{N_{m+2}}$, is an internal iteration of N_{m+2}. □

Lemma 2.5.4. *Every nice sequence $\langle \alpha_0, \ldots, \alpha_m \rangle$ below α can be completed to a nice sequence–*

$$\langle \alpha_0^0, \ldots, \alpha_0^{n_0}, \alpha_1^0, \ldots, \alpha_1^{n_1}, \ldots\ldots, \alpha_m^0, \ldots, \alpha_m^{n_m} \rangle$$

where $\alpha_0^{n_0} = \alpha_0, \alpha_1^{n_1} = \alpha_1, \ldots, \alpha_m^{n_m} = \alpha_m$, such that the embedding k^* associated to the latter sequence is an iteration of N^*.

Proof. We begin with an arbitrary nice sequence $\langle \alpha_0, \ldots, \alpha_m \rangle$, and complete it to a nice sequence–

$$\langle \alpha_0^0, \ldots, \alpha_0^{n_0}, \alpha_1^0, \ldots, \alpha_1^{n_1}, \ldots\ldots, \alpha_m^0, \ldots, \alpha_m^{n_m} \rangle$$

where $\alpha_0^{n_0} = \alpha_0, \alpha_1^{n_1} = \alpha_1, \ldots, \alpha_m^{n_m} = \alpha_m$.

We first extend the sequence below α_0, namely define $\alpha_0^0, \ldots, \alpha_0^{n_0}$.

Denote $N_0^0 = \text{Ult}\,(V, U)$ and $i_0^0 = j_U \colon V \to N_0^0$. Let $\lambda_0^0 = i_0^0\,(g_0)\,(\kappa)$. Let $\alpha_0^0 \leq \alpha_0$ be the first such that $\lambda_0^0 \leq \mu_{\alpha_0^0}$. $\left(\text{cf}\left(\lambda_0^0\right)\right)^V > \kappa$, so actually $\lambda_0^0 = \mu_{\alpha_0^0}$. If $\alpha_0^0 = \alpha_0$, we set $n_0 = 0$ and we are done extending the sequence below α_0. Assume otherwise.

Work in N_0^0 and define there $W_0^0 = i_0^0\,(\mathcal{U})\left(\lambda_0^0\right)$. Let $N_0^1 = \text{Ult}\left(N_0^0, W_0^0\right)$ and $i_0^1 = j_{W_0^0}^{N_0^0} \circ i_0^0 \colon V \to N_0^1$. Define $k_0^1 \colon N_0^1 \to M_{\alpha_0^0+1}$ to be such that for every $f \in V$,

$$k_0^1\left(i_0^1\,(f)\left(\kappa, \lambda_0^0\right)\right) = j_{\alpha_0^0+1}\,(f)\left(\kappa, \mu_{\alpha_0^0}\right)$$

by lemma 2.5.3, k_0^1 is an iterated ultrapower of N_0^1. The measures participating in this iteration lie on measurables below α_0^0 (actually, $k_0^1 = j_{1,\alpha_0^0}^{N_0^1}$). In N_0^1, let $\lambda_0^1 = j_{W_0^0}\left(\lambda_0^0\right)$, and note that λ_0^1 is measurable in N_0^1 above λ_0^1. Thus, λ_0^1 does not participate in the iteration k_0^1, namely $k_0^1\left(\lambda_0^1\right) = \lambda_0^1$. So λ_0^1 is a measurable cardinal in $M_{\alpha_0^0+1}$, and $\left(\text{cf}\left(\lambda_0^1\right)\right)^V > \kappa$. Thus, there exists an index α_0^1, such that $\mu_{\alpha_0^1} = \lambda_0^1$ and $\alpha_0^0 < \alpha_0^1 \leq \alpha_0$. If $\alpha_0^1 = \alpha_0$, we finish extending the sequence below α_0 and set $n_0 = 1$. Assume otherwise. Define in N_0^1 the measure $W_0^1 = i_0^1(\mathcal{U})\left(\lambda_0^1\right)$. Let $N_0^2 = \text{Ult}\left(N_0^1, W_0^1\right)$ and $i_0^2 = j_{W_0^1}^{N_0^1} \circ i_0^1 \colon V \to N_0^2$. Define $k_0^2 \colon N_0^2 \to M_{\alpha_0^1+1}$ in the natural way, namely, for every $f \in V$,

$$k_0^2\left(i_0^2(f)\left(\kappa, \lambda_0^0, \lambda_0^1\right)\right) = j_{\alpha_0^1+1}(f)\left(\kappa, \lambda_0^0, \lambda_0^1\right)$$

and by 2.5.3, k_0^2 is an iterated ultrapower of N^2 with measurables below $\mu_{\alpha_0^1}$. Denote $\lambda_0^2 = j_{W_0^1}^{N_0^1}\left(\lambda_0^1\right) > \lambda_0^1$. Arguing as before, λ_0^2 is measurable in $M_{\alpha_0^1+1}$ with cofinality above κ, and thus, there exists α_0^2 such that $\lambda_0^2 = \mu_{\alpha_0^2}$ and $\alpha_0^0 < \alpha_0^1 < \alpha_0^2 \leq \alpha_0$.

Continue in this fashion, and construct an increasing sequence $\alpha_0^0 < \alpha_0^1 < \ldots \leq \alpha_0$. We argue that the construction stops after finitely many steps. Assume otherwise, and let $\langle \alpha_0^n \colon n < \omega \rangle$ be a strictly increasing sequence of ordinals below α_0, such that for every $n < \omega$,

$$\mu_{\alpha_0} > \mu_{\alpha_0^{n+1}} = \lambda_0^{n+1} = j_{W_0^n}\left(\lambda_0^n\right) > \lambda_0^n = \mu_{\alpha_0^n}$$

and–

$$\mu_{\alpha_0^{n+1}} = \lambda_0^{n+1} = k_0^{n+1}\left(\lambda_0^{n+1}\right) = k_0^{n+1}\left(i_0^{n+1}\left(g_0\right)(\kappa)\right) = j_{\alpha_0^n+1}\left(g_0\right)(\kappa) = j_{U_{\mu_{\alpha_0^n}}}\left(\mu_{\alpha_0^n}\right)$$

let $\alpha_0^* = \sup\{\alpha_0^n : n < \omega\} \leq \alpha_0$. Note that–

$$j_{\alpha_0^*}(g_0)(\kappa) = j_{\alpha_0^0,\alpha_0^*}\left(\mu_{\alpha_0^0}\right) = \sup\{\mu_{\alpha_0^n} : n < \omega\} < \mu_{\alpha_0^*}$$

and thus–

$$\mu_{\alpha_0} = j_{\alpha_0}\left(g_0\right)(\kappa) = j_{\alpha_0^*,\alpha_0}\left(j_{\alpha_0^*}\left(g_0\right)(\kappa)\right) = j_{\alpha_0^*}\left(g_0\right)(\kappa)$$

which contradicts the fact that $j_{\alpha_0^*}\left(g_0\right)(\kappa) < \mu_{\alpha_0^*} \leq \mu_{\alpha_0}$.

Thus, there exists $n_0 < \omega$ and a sequence $\alpha_0^0 < \alpha_0^1 < \ldots < \alpha_0^{n_0} = \alpha_0$ such that for every $n < n_0$,

$$\mu_{\alpha_0^{n+1}} = j_{U_{\mu_{\alpha_0^n}}}\left(\mu_{\alpha_0^n}\right) = j_{\alpha_0^{n+1}}\left(g_0\right)(\kappa)$$

where the last equality follows by induction, since–

$$j_{\alpha_0^{n+1}}(g_0)(\kappa) = j_{\alpha_0^n,\alpha_0^{n+1}}\left(j_{\alpha_0^n}(g_0)(\kappa)\right) = j_{\alpha_0^n,\alpha_0^{n+1}}\left(\mu_{\alpha_0^n}\right) = j_{U_{\alpha_0^n}}\left(\mu_{\alpha_0^n}\right)$$

let us justify the last equality in the above equation. If μ_{α_0} is not a limit of measurables, then $\alpha_0^{n+1} = \alpha_0^n + 1$ and the equation is clear. Otherwise, $\mu_{\alpha_0^n}$ is a limit of measurables. Therefore $\mu_{\alpha_0^{n+1}} = j_{U_{\alpha_0^n}}\left(\mu_{\alpha_0^n}\right)$ is a limit of measurables, and each factor in $j_{\alpha_0^n+1,\alpha_0^{n+1}}$ is an ultrapower embedding with one of them. Thus, each such factor maps $\mu_{\alpha_0^{n+1}}$ to itself.

This finishes the completion of the initial nice sequence below α_0. Let N_0^* be the iterated ultrapower associated to the nice sequence $\langle\alpha_0^0, \ldots \alpha_0^n\rangle$, with a corresponding embedding $i_0^* \colon V \to N_0^*$. Let $k_0^* \colon N_0^* \to M_{\alpha_0+1}$ be defined as follows: for every $f \in V$,

$$k_0^*\left(i_0^*(f)\left(\kappa, \mu_{\alpha_0^0}, \ldots, \mu_{\alpha_0^{n_0}}\right)\right) = j_{\alpha_1}(f)\left(\kappa, \mu_{\alpha_0^0}, \ldots, \mu_{\alpha_0^{n_0}}\right)$$

By lemma 2.5.3, the embedding k_0^* is an iterated ultrapower of N_0^*, and $j_{\alpha_0+1} = k_0^* \circ i_0^*$. All the ultrapowers in k_0^* are taken on measurables below α_0.

Now work over N_0^*, define $\lambda_1^0 = i_0^*\left(g_1\right)\left(\kappa, \mu_{\alpha_0}\right)$. $\lambda_1^0 > \mu_{\alpha_0}$ is measurable in N_0^* and thus is not moved by k_0^*. Also, it has cofinality above κ in V. Let $\alpha_1^0 \leq \alpha_1$ be such that $\lambda_1^0 = \mu_{\alpha_1^0}$. If

$\lambda_1^0 = \mu_{\alpha_1}$, we set $n_1 = 0$ and move on to extend the sequence below α_2. Assume otherwise. Let $W_1^0 = i_0^*(\mathcal{U})(\lambda_1^0)$. Let $N_1^0 = \mathrm{Ult}(N_0^*, W_1^0)$, and $i_1^0 = j_{W_1^0}^{N_0^*} \circ i_0^*$. Let $k_1^0 \colon N_0^1 \to M_{\alpha_1^0+1}$ be the natural embedding, and continue the construction as above. It will stop after finitely many steps.

By repeating the same argument for $\alpha_2, \ldots, \alpha_m$, we generate the desired completion of $\langle \alpha_0, \ldots, \alpha_m \rangle$.

□

Remark 2.5.5. *For every $0 \le i \le m$, μ_{α_i} appears in the Prikry sequence of $\mu^*_{\alpha_i} = k_{\alpha_i}(\mu_{\alpha_i})$ in $M[H]$. Note that in the above proof, the completion below μ_{α_i}, namely the sequence $\langle \alpha_0^0 \ldots, \alpha_i^{n_i} \rangle$, is a subsequence of the Prikry sequence of $\mu^*_{\alpha_i}$ below μ_{α_i}. In lemma 2.5.8 we will prove that this subsequence is actually a segment in this Prikry sequence.*

Lemma 2.5.6. *Assume that $\alpha < \kappa^*$. Denote $\bar{\mu} = \sup\{\mu_{\alpha'} : \alpha' < \alpha\}$. Let $\lambda > \bar{\mu}$ be an inaccessible cardinal in M_α. Then $(cf(\lambda))^V > \kappa$.*

Proof. Let us first consider the case where there is no $\beta < \alpha$ and $\lambda' < \lambda$ such that $j_{\beta,\alpha}(\lambda') = \lambda$. Let $\langle \alpha_0, \ldots, \alpha_m \rangle$ be a nice sequence below α such that $\lambda = j_\alpha(h)(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_m})$ for some function $h \in V$. We can assume that the sequence in complete as in lemma 2.5.4, and so $k^* \colon N^* \to M_\alpha$ is an internal iterated ultrapower. Denote–

$$\lambda^* = i^*(h)(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_m})$$

and note that $k^*(\lambda^*) = \lambda$. It suffices to prove that $\lambda^* = \lambda$, since every inaccessible above κ in a finite iteration of V has cofinality $> \kappa$ in V. Assume that $\lambda^* < \lambda$. Because λ^* is inaccessible in N^*, λ^* is one of the measurables participating in the iteration k, namely $\lambda^* = \mu_\beta$ for some $\beta < \alpha$. Since $\lambda^* > \mu_{\alpha_m}$, $\beta > \alpha_m$. Then–

$$\begin{aligned} \lambda &= k^*(\lambda^*) \\ &= \left(j_{\beta,\alpha} \circ j_{\alpha_m+1,\beta} \circ j_{\alpha_{m-1}+1,\alpha_m} \circ \ldots \circ j_{\alpha_0+1,\alpha_1} \circ j_{1,\alpha_0}\right)(\lambda^*) \\ &= j_{\beta,\alpha}(\lambda^*) \end{aligned}$$

where we used the fact that $\lambda^* = \mu_\beta$ is inaccessible in N^* above μ_{α_m}, and thus is fixed by ultrapowers below μ_{α_m} and by $j_{\alpha_m+1,\beta}$. It follows that there exists $\beta < \alpha$ and $\lambda^* < \lambda$ such that $j_{\beta,\alpha}(\lambda^*) = \lambda$, which is a contradiction.

Let us now take care of the case where, for some $\beta < \alpha$ and $\lambda_0 < \lambda$, $j_{\beta,\alpha}(\lambda_0) = \lambda$. Let $\beta < \alpha$ be the least such that such λ_0 exists. Since λ_0 is inaccessible in M_β and $\lambda_0 < j_{\beta,\alpha}(\lambda_0)$, λ_0 is one of the measurables participating in the iteration $j_{\beta,\alpha}$. Thus, $\lambda_0 = \mu_{\gamma_0}$, for some $\gamma_0 < \alpha$.

Denote $\lambda_1 = j_{U_{\mu_{\gamma_0}}}(\mu_{\gamma_0})$. This is an inaccessible cardinal in M_{γ_0+1}. Let us argue that $(\mathrm{cf}(\lambda_1))^V > \kappa$.

Pick a complete nice sequence $\langle \alpha_0, \ldots, \alpha_m \rangle$ below $\gamma_0 + 1$ such that, for some function $h \in V$,

$$\lambda_1 = j_{\gamma_0+1}(h)(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_m})$$

we can assume that $\alpha_m = \gamma_0$ (else, add it. The sequence will remain complete since there is no $\lambda' < \lambda_0$ and $\gamma' \leq \gamma_0$ such that $j_{\gamma',\gamma_0+1}(\lambda') = \lambda_0$). Let N^* be the associated finite iteration, with an embedding $i^* \colon V \to N^*$. let $k^* \colon N^* \to M_{\gamma_0+1}$ be the corresponding iterated ultrapower such that $k^* \circ i^* = j_{\gamma_0+1}$. Denote $\lambda_1^* = i^*(h)(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_m})$. Then $k^*(\lambda_1^*) = \lambda_1$. Let us argue that $\lambda_1^* = \lambda_1$. Assume that $\lambda_1^* < \lambda_1$. Then λ_1^*, which is measurable in N^*, is one of the measurables participating in k^*. Note that–

$$\lambda_1 = k^*(\lambda_1^*) = j_{\alpha_m+1,\gamma_0+1} \circ j_{\alpha_{m-1}+1,\alpha_m} \circ j_{\alpha_{m-2}+1,\alpha_{m-1}} \circ \ldots \circ j_{1,\alpha_0}(\lambda_1^*)$$

but $\alpha_m = \gamma_0$, so $j_{\alpha_m+1,\gamma_0+1}$ is the identity. So $\lambda_1^* < \mu_{\alpha_m} = \mu_{\gamma_0}$. μ_{γ_0} is already a non-measurable inaccessible in N^* (since we started from a complete nice sequence which includes it), and thus $k^*(\mu_{\gamma_0}) = \mu_{\gamma_0}$. Namely $j_{U_{\gamma_0}}(\mu_{\gamma_0}) = \lambda_1 = k^*(\lambda_1^*) < \mu_{\gamma_0}$, a contradiction.

Thus $(\mathrm{cf}(\lambda_1))^V > \kappa$. If $\lambda_1 = \lambda$, we are done. Else, $\lambda_1 < \lambda$ is inaccessible in M_{γ_0+1}, and is mapped via $j_{\gamma_0+1,\alpha}$ to λ. Hence, arguing as before, $\lambda_1 \leq \bar{\mu}$ is one of the measurables participating in the iterated ultrapower $j_{\gamma_0+1,\alpha}$. Therefore, there exists $\gamma_1 \in (\gamma_0, \alpha)$ such that $\lambda_1 = \mu_{\alpha_{\gamma_1}}$. Denote $\lambda_2 = j_{U_{\mu_{\gamma_1}}}(U_{\mu_{\gamma_1}}) > \lambda_1$. As above, $(\mathrm{cf}(\lambda_2))^V > \kappa$. If $\lambda_2 = \lambda$, we are done. Assume otherwise, and continue in this fashion.

Let us argue that the process stops after finitely many steps. Assume otherwise. Then we have constructed an ω-sequence of ordinals below α, $\langle \gamma_n \colon n < \omega \rangle$, and an increasing sequence–

$$\lambda_0 = \mu_{\gamma_0} < \lambda_1 = \mu_{\gamma_1} < \lambda_2 = \mu_{\gamma_2} < \ldots < \lambda$$

such that, for every $n < \omega$, $\lambda_{n+1} = \mu_{\gamma_{n+1}} = j_{U_{\mu_{\gamma_n}}}(\mu_{\gamma_n})$. Denote $\gamma^* = \sup\{\gamma_n \colon n < \omega\}$ (possibly $\gamma^* = \alpha$). Let $\lambda^* = \sup\{\lambda_n \colon n < \omega\}$. Then–

$$j_{\gamma^*,\alpha}(\lambda^*) = \lambda$$

however, $j_{\gamma^*,\alpha}(\lambda^*) = \lambda^*$: if $\gamma^* = \alpha$ this is clear. Else, note that μ_{γ^*} is chosen strictly above $\sup\{\mu_\xi \colon \xi < \gamma^*\} = \lambda^*$. Therefore, the critical point of $j_{\gamma^*,\alpha}$ is above λ^*, and $j_{\gamma^*,\alpha}(\lambda^*) = \lambda^*$.

It follows that $\lambda^* = \lambda$. But $\lambda^* \leq \bar{\mu}$ (equality may hold if $\gamma^* = \alpha$), contradicting the fact that $\lambda > \bar{\mu}$. $\square$

We now return to our context, and assume that $\langle M_\alpha \colon \alpha \leq \kappa^* \rangle$ is the iteration described in the previous section, with the same notations. We can first simplify the definition of the critical points μ_α:

Corollary 2.5.7. *Assume that $\alpha < \kappa^*$. Let $\bar{\mu} = \sup\{\mu_{\alpha'} \colon \alpha' < \alpha\}$.*
If α is successor, μ_α is the first measurable above $\bar{\mu}$ in M_α.
If α is limit and $(cf(\alpha))^V \leq \kappa$, then μ_α is the first measurable above $\bar{\mu}$ in M_α.
If α is limit and $(cf(\alpha))^V > \kappa$, then μ_α is the first measurable in M_α which is greater or equal to $\bar{\mu}$.

Proof. If $\bar{\mu}$ is measurable in M_α and $(\mathrm{cf}(\alpha))^V > \kappa$, then $(\mathrm{cf}(\bar{\mu}))^V > \kappa$ and thus $\mu_\alpha = \bar{\mu}$ by the definition. Else, μ_α is chosen to be the least measurable in M_α above $\bar{\mu}$ with cofinality above κ in V, which is, by the previous lemma, the least measurable above $\bar{\mu}$ in M_α. $\square$

Lemma 2.5.8. *Assume that $\alpha < \kappa^*$ and λ appears after μ_α in the Prikry sequence of $\mu^* = k_\alpha(\mu_\alpha)$. Then $\lambda = j_{U_{\mu_\alpha}}(\mu_\alpha)$.*

Proof. Since $j_{U_{\mu_\alpha}}(\mu_\alpha)$ is measurable in $M_{\alpha+1}$ above $\bar{\mu}_{\alpha+1} = \sup\{\mu_{\alpha'} \colon \alpha' \leq \alpha\}$, it follows, by lemma 2.5.6, that–

$$\left(\mathrm{cf}\left(j_{U_{\mu_\alpha}}(\mu_\alpha)\right)\right)^V > \kappa$$

Thus there exists $\beta > \alpha$ such that $j_{U_{\mu_\alpha}}(\mu_\alpha) = \mu_\beta$, and appears in the Prikry sequence of $k_\beta(\mu_\beta) = j_{\beta,\kappa^*}(j_{\alpha,\beta}(\mu_\alpha)) = \mu^*$.

Let us prove now that $j_{U_{\mu_\alpha}}(\mu_\alpha) = \mu_\beta$ is the immediate successor of μ_α in the Prikry sequence of μ^*.

Assume, for contradiction, that $\mu_\alpha < \lambda < j_{U_{\mu_\alpha}}(\mu_\alpha)$, and λ appears after μ_α in the Prikry sequence of μ^*. Assume that $\lambda = j_{\alpha+1}(g)(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k}, \mu_\alpha)$, for some $g \in V$ and $\alpha_0 < \ldots < \alpha_k < \alpha$. Assume also that $h \in V$ is a function such that $\mu_\alpha = j_\alpha(h)(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k})$ for the same $\alpha_0 < \ldots < \alpha_k < \alpha$ (this can always be arranged by changing the sequence $\langle \alpha_0, \ldots, \alpha_k \rangle$). Then–

$$j_{\alpha+1}(g)(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k}, \mu_\alpha) < j_{\alpha+1}(h)(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k})$$

so we may assume that for every $\xi, \nu_0, \ldots, \nu_k, \eta$, below κ, $g(\xi, \nu_0, \ldots, \nu_k, \eta) < h(\xi, \nu_0, \ldots, \nu_k)$. Assume also that μ_α is the n-th element in the Prikry sequence of μ^*. In $V[G]$, let $\lambda(\xi)$ be the $(n+1)$-th element in the Prikry sequence of $h(\xi, \vec{\mu}(\xi))$, so that $[\xi \mapsto \lambda(\xi)]_W = \lambda$.

Assume that the sequence $\langle \alpha_0, \ldots, \alpha_k \rangle \subseteq \alpha$ is nice (else, add more coordinates). Now apply the Multivariable Fusion Lemma. For every $\langle \xi, \vec{\nu} \rangle$, let–

$$\begin{aligned} e\left(\xi, \vec{\nu}\right) = \{ r \in P \setminus \nu_k + 1 \colon & \text{ there exists a bounded subset } A \subseteq h\left(\xi, \vec{\nu}\right) \text{ such that} \\ & r \text{ forces that the } (n+1)\text{-th element in the Prikry sequence of } h\left(\xi, \vec{\nu}\right) \\ & \text{belongs either to } A \text{ or to the club of closure points of the function} \\ & \eta \mapsto g\left(\xi, \vec{\nu}, \eta\right) \} \end{aligned}$$

We argue that $e\left(\xi, \vec{\nu}\right)$ is $\leq^*$ dense open above any condition which forces that $\underset{\sim}{\vec{\mu}}(\xi) = \vec{\nu}$. Let $p \in P \setminus \nu_k + 1$ be such a condition. Denote for simplicity $h = h\left(\xi, \vec{\nu}\right)$. Direct extend $p \restriction_h$ such that it decides the length of t^p_h; if the length is $\geq (n+1)$, direct extend $p \restriction_h$ further, so that it forces that there exists a bounded subset $A \subseteq h$ such that the $(n+1)$-th element in the Prikry sequence of h belongs to it. Finally, shrink $\underset{\sim}{A}^p_h$ by intersecting with the club of closure points of the function which maps each $\eta < h$ to $g\left(\xi, \vec{\nu}, \eta\right)$. The condition obtained this way indeed belongs to $e\left(\xi, \vec{\nu}\right)$.

Now, fix $p \in G$ and a C-tree T such that for every $\langle \xi, \vec{\nu} \rangle \in T$ which is admissible for p,

$$\begin{aligned} & \left(p^\frown \langle \xi, \vec{\nu} \rangle\right) \restriction_{\nu_k+1} \Vdash \text{ there exists a bounded subset } A \subseteq h\left(\xi, \vec{\nu}\right) \text{ such that} \\ & p^\frown \langle \xi, \vec{\nu} \rangle \setminus \left(\nu_k + 1\right) \text{ forces that the } (n+1)\text{-th element in the Prikry sequence} \\ & \text{of } h\left(\xi, \vec{\nu}\right) \text{ belongs either to } A \text{ or to the club of closure points of the function} \\ & \eta \mapsto g\left(\xi, \vec{\nu}, \eta\right) \} \end{aligned}$$

Let $\underset{\sim}{A}\left(\xi, \vec{\nu}\right)$ be a P_{ν_k+1}-name for the set A above, and set–

$$A^*\left(\xi, \vec{\nu}\right) = \{ \gamma < h\left(\xi, \vec{\nu}\right) : \exists r \geq p^\frown \langle \xi, \vec{\nu} \rangle \restriction_{\nu_k+1},\ r \Vdash \gamma \in \underset{\sim}{A}\left(\xi, \vec{\nu}\right) \}$$

It follows that for a set of ξ-s in W, $\lambda(\xi)$ either belongs to $A^*\left(\xi, \vec{\mu}(\xi)\right)$ or to the club of closure points of $\eta \mapsto g\left(\xi, \vec{\mu}(\xi), \eta\right)$.

However, it cannot hold that for a set of ξ-s in W, $\lambda(\xi) \in A^*\left(\xi, \vec{\mu}(\xi)\right)$. Indeed assume otherwise. Denote–

$$A^* = j_\alpha\left(\langle \xi, \vec{\nu} \rangle \mapsto A^*\left(\xi, \vec{\nu}\right)\right)(\kappa, \mu)$$

then A^* is bounded in μ_α, and, under the above assumption, $\lambda \in k_\alpha\left(A^*\right) = A^* \subseteq \mu_\alpha$, which is a contradiction.

Thus, in $M[H]$, λ is a closure point of $\eta \mapsto j_W(g)\left(\kappa, \vec{\mu}, \eta\right)$. Recall that $\mu_\alpha < \lambda$, and thus $j_W(g)\left(\kappa, \vec{\mu}, \mu_\alpha\right) < \lambda = j_\alpha(g)\left(\kappa, \vec{\mu}, \mu_\alpha\right)$, which is a contradiction. $\square$

Corollary 2.5.9. *Let $\alpha < \kappa^*$ and denote $\mu^* = k_\alpha(\mu_\alpha)$. Then the Prikry sequence of μ^* in $M[H]$ has a final segment of the form–*

$$\langle \mu_{\alpha_0}, \mu_{\alpha_1}, \mu_{\alpha_2}, \ldots, \mu_{\alpha_n}, \ldots \rangle$$

where $\alpha_0 = \alpha$, and for every $n < \omega$, $\mu_{\alpha_{n+1}} = j_{U_{\mu_{\alpha_n}}}(\mu_{\alpha_n})$. Furthermore, the above sequence belongs to V, namely $(cf(\mu^*))^V = \omega$.

Proof. The first part follows immediately from the previous lemma. Let us concentrate on the second part. Assume that there is no $\beta < \alpha_0$ and $\mu < \mu_{\alpha_0}$ such that $j_{\beta,\alpha}(\mu) = \mu_{\alpha_0}$ (if there is, replace μ_{α_0} with the least such μ). Let $\beta_0, \ldots, \beta_k$ be a complete nice sequence such that $\mu_{\alpha_0} = j_{\alpha_0}(h)(\beta_0, \ldots, \beta_k)$ for some $h \in V$. It follows that the sequence $\langle \beta_0, \ldots, \beta_k, \alpha_0, \alpha_1, \ldots, \alpha_n \rangle$ is complete, for every $n < \omega$. Then, for every $n < \omega$, a finite iteration $\langle N_i : i \leq n+1 \rangle$ can be defined as in the beginning of this section. If $f \in V$ is a function such that $U_{\mu_{\alpha_0}} = j_{\alpha_0}(f)(\kappa, \mu_{\beta_0}, \ldots, \mu_{\beta_k})$, then the sequence $\langle N_i : i < \omega \rangle$ is definable in V, since each step above the first k-many steps in the iteration, uses a measure represented by f. Because each sequence $\langle \beta_0, \ldots, \beta_k, \alpha_0, \ldots, \alpha_n \rangle$ is complete, the sequence $\langle \mu_{\alpha_0}, \mu_{\alpha_1}, \ldots, \mu_{\alpha_n}, \ldots \rangle$ is a final segment of the sequence of critical points in the iteration $\langle N_i : i < \omega \rangle$, and thus belongs to V. □

Remark 2.5.10. *We would like to emphasize the point that the characterization of Prikry sequences given in the previous corollary is given only up to some finite initial segment. Let us denote $\mu = \mu_0$, which is the first measurable above κ in M_U, and $\mu^* = k_0(\mu_0)$ which is the first measurable above κ in M. We argue that the Prikry sequence of μ^* in $M[H]$ may have any prescribed finite initial segment $t \in [\mu]^{<\omega}$. We use those notations only in the following claim:*

Claim 2.5.11. *For every finite, increasing sequence $t \in [\mu]^{<\omega}$, there exists a condition $p \in P_\kappa$ which forces that t is an initial segment of the Prikry sequence of μ^* in $M[H]$.*

Proof. Assume that $\xi \mapsto t(\xi)$ is a function in V such that $[\xi \mapsto t(\xi)]_U = t$. For each $\xi < \kappa$, let $s(\xi)$ be the first measurable strictly above ξ. Then $[\xi \mapsto s(\xi)]_U = \mu$. Since $\mu > \max(t)$, we can assume that for every $\xi < \kappa$, $\max(t(\xi)) < s(\xi)$.

Note that the set $\{s(\xi) : \xi < \kappa\} \cap \lambda$ is nonstationary in any inaccessible $\lambda \leq \kappa$: This is clear if λ is not a limit of measurables. If it is, $\{s(\xi) : \xi < \kappa\}$ is disjoint to the club of limit points of $\Delta = \{\alpha < \kappa : \alpha \text{ is measurable}\}$ below λ.

Now, let us define a condition $p \in P_\kappa$, with $\text{supp}(p) = \{s(\xi) : \xi < \kappa\}$. We first choose a set $X \in U$ on which the function $\xi \mapsto s(\xi)$ with domain X is injective. Note that by normality of U, every function is either one-to-one or constant modulo U, so such a set $X \in U$ exists.

Set, for a given $\xi \in X$, $p(s(\xi)) = \langle t(\xi), s(\xi) \setminus (\max(t(\xi)) + 1) \rangle$. This is forced by any condition in $P \restriction_{s(\xi)}$ to be a legitimate element of $\underset{\sim}{Q}_{s(\xi)}$. Note that the definition makes sense since $\xi \mapsto s(\xi)$ is injective on X. The condition $p \in P_\kappa$ defined in this way forces that the Prikry sequence of μ^* starts with t: Indeed, in $V[G]$,

$$\{\xi < \kappa : t(\xi) \text{ is an initial segment of the Prikry sequence of } s(\xi)\} \supseteq X \in W$$

thus, in $M[H]$, $[\xi \mapsto t(\xi)]_W$ is an initial segment of the Prikry sequence of the measurable cardinal $[\xi \mapsto s(\xi)]_W$. But by lemma 2.3.2,

$$[\xi \mapsto t(\xi)]_W = k\left([\xi \mapsto t(\xi)]_U\right) = k(t) = t$$

and clearly–

$$[\xi \mapsto s(\xi)]_W = \mu^*$$

so in $M[H]$, t is an initial segment of the Prikry sequence added to μ^*. □

Let us prove now that for every measurable μ^* above κ in M, μ^* has the form $k_\alpha(\mu_\alpha)$ for some $\alpha < \kappa^*$. In particular, in the light of corollary 2.5.9, $(\text{cf}(\mu^*))^V = \omega$.

Lemma 2.5.12. *Assume that $\mu^* \in (\kappa, \kappa^*)$ is measurable in M. Then $\mu^* = k_\alpha(\mu_\alpha)$ for some $\alpha < \kappa^*$.*

Proof. Let $\beta < \kappa^*$ be the first such that, for some $\mu \leq \mu^*$, $\mu^* = k_\beta(\mu)$. Then μ is measurable in M_β. β is either 0 or a successor by its minimality. Assume first that $\beta = \alpha + 1$. $\mu = \mu_\alpha$ cannot hold since μ_α is not measurable in $M_{\alpha+1}$. If $\mu < \mu_\alpha$ then $j_{\alpha,\beta}(\mu) = \mu$, contradicting the minimality. Thus assume that $\mu > \mu_\alpha = \bar{\mu}_\beta = \sup\{\mu_{\beta'} : \beta' < \beta\}$. Recall that μ is measurable in M_β. By lemma 2.5.6, $(\text{cf}(\mu))^V > \kappa$. Therefore, for some $\gamma \in [\beta, \kappa^*)$, $\mu = \mu_\gamma$. Hence $k_\gamma(\mu_\gamma) = \mu^*$.

If $\beta = 0$ then μ is measurable in M_0 above κ and below κ^*, and clearly $(\text{cf}(\mu))^V > \kappa$. So, again, there exists $\gamma < \kappa^*$ such that $\mu = \mu_\gamma$, and $k_\gamma(\mu_\gamma) = \mu^*$. □

Corollary 2.5.13. *Assume that $\alpha < \kappa^*$ is limit, and denote $\bar{\mu} = \sup\{\mu_{\alpha'} : \alpha' < \alpha\}$. Assume that $\bar{\mu}$ is measurable in M_α. Then $(cf(\alpha))^V$ is either ω or κ^+. In the former case, $\bar{\mu}$ is measurable in M. In the latter case, $\bar{\mu} = \mu_\alpha$ is a non-measurable inaccessible cardinal in M. Moreover:*

1. *If $\bar{\mu}$ is not measurable in M_α or $(cf(\alpha))^V > \kappa$, μ_α is the first measurable $\geq \bar{\mu}$ in M_α (this includes the case where α is successor, since, in this case, $\mu_{\alpha-1}$ is not measurable in M_α).*

2. *Else, $\bar{\mu}$ is measurable in M_α and $(cf(\alpha))^V = \omega$, and then $\mu_\alpha = \bar{\mu}$.*

Proof. Assume that $\bar{\mu}$ is measurable in M_α. If $(\mathrm{cf}(\alpha))^V \leq \kappa$, then $\bar{\mu} < \mu_\alpha$, so $\bar{\mu} = k_\alpha(\bar{\mu})$ is measurable in M. By the previous lemma, $\bar{\mu} = k_\gamma(\mu_\gamma)$ for some $\gamma < \kappa^*$. By corollary 2.5.9, $(\mathrm{cf}(\bar{\mu}))^V = \omega$. Hence $(\mathrm{cf}(\alpha))^V = \omega$. □

Remark 2.5.14. *Recall that $j_W(\mathcal{U}) \setminus \kappa \in M$ is sufficient for the definability of $j_W \restriction_V$ over V. Let us argue that it is not necessary.*

For every measurable $\eta < \kappa$, let $\langle s^n(\eta) \colon n < \omega \rangle$ be the increasing enumeration of the first ω-many measurables above η which carry at least η-many normal measures of Mitchell order 0. For each such η and $n < \omega$, let $\vec{F}(s^n(\eta))$ be an enumeration for all the normal measures of order 0 on $s^n(\eta)$. Fix an unbounded nonstationary subset $X \subseteq \Delta$ such that for every $\eta \in X$ and $n < \omega$, $s^n(\eta) \notin X$. Let P be the forcing notion which uses, at stage $s^n(\eta)$ where $\eta \in X$ and $n < \omega$, the measure which extends $\left(\vec{F}(s^n(\eta))\right)(\eta_n)$. Here, $\eta_n < \eta$ is the n-th element in the Prikry sequence of η in $M[H]$. For every other measurable, use the measure chosen first with respect to a prescribed well order of V_κ.

Pick a generic set $G \subseteq P$ such that G contains a condition p such that $X \subseteq supp(p)$, but for every $\xi \in X$, $p \restriction_\xi \Vdash t^p_\xi = \langle\rangle$.

Then $j_W(\mathcal{U}) \setminus \kappa \notin M$, since the measures used in $j_W(P)$ on M-measurables above κ code the Prikry sequences of all the measurables in $j_W(X) \setminus \kappa$.

However, $j_W \restriction_V$ is definable in V: Assume that $\alpha < \kappa^*$. If there is no $\eta \in j_\alpha(X)$ and $n < \omega$ such that $\mu_\alpha = s^n(\eta)$, U_{μ_α} is the first measure on α with respect to the image under j_α of the prescribed well order on V_κ. Otherwise, assume that $\eta \in j_\alpha(X)$, $n < \omega$ and $\mu_\alpha = s^n(\eta)$. Denote $\eta^* = k_\alpha(\eta)$, so that $k_\alpha(\mu_\alpha) = s^n(\eta^*)$. Let $\beta = \beta_0 < \kappa^*$ be the least such that $k_\beta(\mu_\beta) = \eta^*$. We argue that the Prikry sequence of η^* in $M[H]$ is the sequence of critical points taken by iteration U_{μ_β} ω-many times over M_β. This will follow once we prove that μ_β is the first element in the Prikry sequence of η^*, and this is true since $\eta^* \in j_W(X)$ and there exists a condition $p \in G$ which forces that $t^p_\xi = \langle\rangle$ for every $\xi \in X$. Thus, we can assume that $\langle \mu_{\beta_0}, \mu_{\beta_1}, \ldots, \mu_{\beta_n}, \ldots \rangle$ is the Prikry sequence of η^* in $M[H]$.

Recall that $k_\alpha(U_{\mu_\alpha}) = j_W(\mathcal{U})(k_\alpha(\mu_\alpha))$; by the definition of the forcing, $j_W(\mathcal{U})(k_\alpha(\mu_\alpha))$ is the Prikry forcing taken with the measure–

$$\left(j_W\left(\vec{F}\right)(k_\alpha(\mu_\alpha))\right)(\mu_{\beta_n})$$

thus, U_{μ_α} can be computed in V as follows: first, calculate over M_{β_0} (which is already definable in V by induction) the sequence $\langle \mu_{\beta_n} : n < \omega \rangle$, which are the critical points in the iteration of length ω with $U_{\mu_{\beta_0}}$ over M_{β_0} (here, U_{μ_β} is the least measure with respect to the image under j_β of the prescribed well order on V_κ); then, compute $U_{\mu_\alpha} = \left(j_\alpha\left(\vec{F}\right)(\mu_\alpha)\right)(\mu_{\beta_n})$.

2.6 Application: Unique Normal Measure on a Strongly Compact Cardinal which is the Least Measurable

Kanamori [14] asked whether a strongly compact can carry a unique normal measure.

Assuming linearity of the Mitchell order, the least measurable cardinal κ which is a limit of strongly compacts is such: by a result of Menas, [19], κ is strongly compact. It carries a unique normal measure, since otherwise, by linearity of the Mitchell order, there exists an ultrapower with a normal measure U on κ such that κ is still measurable in M_U (just take U of Mitchell order above 0). By elementarity, every strongly compact cardinal below κ remains such in M_U. Thus, U concentrates on the set of measurable limits of strongly compacts, contradicting the minimality of κ.

The next step would be to ask whether the least strongly compact can be the least measurable, and carry a unique normal measure.

Wooding and Goldberg proved that this is consistent, assuming the Ultrapower Axiom and a measurable limit of supercompact cardinals.

In [8] the same result was proved starting from a single supercompact and assuming linearity of the Mitchell order (which is a consequence of the Ultrapower Axiom, see, for example, [11]). We present here the same argument that appears in [8], under simpler settings - assuming that GCH holds as well.

Theorem 2.6.1. *(Gitik, K., [8]) Assume that κ is a supercompact cardinal, GCH holds, and the Mitchell order is linear. Then it is consistent that κ is strongly compact, the only measurable cardinal, and carries a unique normal measure.*

Proof. By chopping the universe if necessary, we can assume also that κ is the last measurable cardinal. Denote by V the chopped universe. GCH still holds in V, and the Mitchell order remains linear. Force over V with the nonstationary support iteration of Prikry forcings, P_κ. Let $G \subseteq P_\kappa$ be generic over V. Then, by the last sections, there exists a unique normal measure on κ in $V[G]$. Every measurable cardinal of V changes cofinality, and no new measurables are generated. Thus, κ is the only measurable cardinal. It thus suffices to prove that κ is strongly compact in $V[G]$. This follows from the next lemma. □

Lemma 2.6.1. *Let $G \subseteq P_\kappa$ be generic over V. Assume that in V, κ is supercompact, and the last measurable cardinal. Then κ is strongly compact in $V[G]$.*

Proof. Assume that $\lambda > \kappa^+$ and $\mathrm{cf}(\lambda) \in [\kappa, \lambda)$. It suffices to prove that there exists a fine measure on $\mathcal{P}_\kappa(\lambda)$ in $V[G]$. Let $\mathcal{U}$ be a fine, normal measure on $\mathcal{P}_\kappa(\lambda)$ in V, with a corresponding elementary embedding $j_\mathcal{U} \colon V \to M_\mathcal{U}$.

For each $p \in G$, $\kappa \notin \mathrm{supp}\,(j_\mathcal{U}(p))$. This holds, since given a club $C \subseteq \kappa$ disjoint to $\mathrm{supp}(p)$, $j_\mathcal{U}(C) \cap \kappa = C$ is unbounded in κ, and thus $\kappa \in j_\mathcal{U}(C)$.

In $M_\mathcal{U}$, factor $j_\mathcal{U}(P_\kappa) = P_\kappa * \underset{\sim}{Q}_\kappa * \underset{\sim}{P}_{>\kappa}$. $G \subseteq P_\kappa$ is generic over $M_\mathcal{U}$. Also, over $M_\mathcal{U}[G]$,

$$0_{\underset{\sim}{Q}_\kappa} \Vdash \text{ the conditions } \langle j_\mathcal{U}(p) \setminus \kappa \colon p \in G \rangle \text{ are pairwise } \leq^* -\text{compatible.}$$

this holds, since for any $p, q \in G$, there exists $r \in G$ such that $p, q \leq r$. Therefore, there is $\alpha < \kappa$ such that for every $x \geq \alpha$, $r \restriction_{x+1} \Vdash r \setminus x \geq^* p \setminus x, q \setminus x$. In particular, $r \restriction_{x+1}$ forces that $p \setminus x, q \setminus x$ are $\leq^*$-compatible, for every $x < \kappa$. By applying $j_\mathcal{U}$ and taking $x = \kappa$, it follows that $j_\mathcal{U}(r) \restriction_{\kappa+1} = r^\frown 0_{\underset{\sim}{Q}_\kappa}$ forces that $j_\mathcal{U}(p) \setminus \kappa, j_\mathcal{U}(q) \setminus \kappa$ are $\leq^*$-compatible.

It follows that there exists a $P_{\kappa+1}$-name, $\underset{\sim}{s}$, for a condition which extends, in the direct extension order, all the conditions $\langle j_\mathcal{U}(p) \colon p \in G \rangle$ (note that $|G| = \kappa^+$ and the direct extension order is closed enough).

We claim that the direct extension order of $j_\mathcal{U}(P) \setminus \kappa$ is more than λ^+-closed. The reasons are as follows: First, in $M_\mathcal{U}$, there are no measurables in the interval (κ, λ), since κ is the last measurable in V and $V \vDash {}^\lambda M_\mathcal{U} \subseteq M_\mathcal{U}$. Second, λ is not measurable in $M_\mathcal{U}$ since $\mathrm{cf}(\lambda) < \lambda$ in V. Finally, $\lambda^+ = (\lambda^+)^{M_\mathcal{U}}$ cannot be measurable in $M_\mathcal{U}$.

Now, note that, in V, $|j_\mathcal{U}(\kappa^+)| = \lambda^+$, since $\mathrm{cf}(\lambda) \geq \kappa$. Assume that $\langle \underset{\sim}{E}(\alpha) \colon \alpha < \lambda^+ \rangle$ is forced, over $M_\mathcal{U}[G]$, by $0_{\underset{\sim}{Q}_\kappa}$, to be the list of all $\leq^*$-dense open subsets of $j_\mathcal{U}(P) \setminus \kappa$ above $\underset{\sim}{s}$.

We construct, in $V[G]$, a sequence of $\underset{\sim}{Q}_\kappa$–names $\langle s_\alpha \colon \alpha < \lambda^+\rangle$, such that $0_{\underset{\sim}{Q}_\kappa}$ forces that the sequence consists of conditions in $j_{\mathcal{U}}(P) \setminus \kappa$, $\underset{\sim}{s}_0 \geq^* \underset{\sim}{s}$, and the sequence is increasing with respect to direct extensions. Moreover, each $\underset{\sim}{s}_\alpha \in \underset{\sim}{E}(\alpha)$. Finally, Let us define $\mathcal{U}^* \in V[G]$ as follows. For every P_κ–name for a subset of $\mathcal{P}_\kappa(\lambda)$, $\underset{\sim}{X}$, $\left(\underset{\sim}{X}\right)_G \in \mathcal{U}^*$ if and only if there exists $p \in G$, $\alpha < \lambda^+$ and a P-name $\underset{\sim}{A}$, such that–

$$p^\frown \langle \langle\rangle, \underset{\sim}{A}\rangle^\frown \underset{\sim}{s}_\alpha \Vdash j''_{\mathcal{U}}\lambda \in j\left(\underset{\sim}{X}\right)$$

We prove that $\mathcal{U}^*$ is a fine measure which extends $\mathcal{U}$. First, we prove that $\mathcal{U} \subseteq \mathcal{U}^*$. Assume that $X \in \mathcal{U}$. Then for any condition $p \in G$, $\{x \in \mathcal{P}_\kappa(\lambda) \colon p \Vdash \check{x} \in \check{X}\} \in U$, namely $j_{\mathcal{U}}(p) \Vdash j''_{\mathcal{U}}\lambda \in j_{\mathcal{U}}\left(\check{X}\right)$. So, for each name $\underset{\sim}{A}$ and $\alpha < \lambda^+$,

$$p^\frown \langle \langle\rangle, \underset{\sim}{A}\rangle^\frown s_\alpha \Vdash j''_{\mathcal{U}}\lambda \in j\left(\underset{\sim}{X}\right)$$

hence $X \in \mathcal{U}^*$. In particular, it follows that $\mathcal{U}^*$ is fine.

$\mathcal{U}^*$ is closed under intersection of finitely many sets, since any pair of conditions of the form $p^\frown \langle \langle\rangle, \underset{\sim}{A}\rangle^\frown \underset{\sim}{s}_\alpha$ and $q^\frown \langle \langle\rangle, \underset{\sim}{B}\rangle^\frown \underset{\sim}{s}_\beta$, where $p, q \in G$, $\underset{\sim}{A}, \underset{\sim}{B}$ are names and $\alpha, \beta < \lambda^+$, are compatible.

Assume now that $\left(\underset{\sim}{X}\right)_G \in \mathcal{U}^*$, $p \in G$ is a condition and $\underset{\sim}{Y}$ a P_κ-name such that $p \Vdash \underset{\sim}{X} \subseteq \underset{\sim}{Y}$. Then $j_{\mathcal{U}}(p) \Vdash j\left(\underset{\sim}{X}\right) \subseteq j\left(\underset{\sim}{Y}\right)$. There are $q \in G$, $\alpha < \lambda^+$ and $\underset{\sim}{A}$, such that $q^\frown \langle \langle\rangle, \underset{\sim}{A}\rangle^\frown s_\alpha \Vdash j''_{\mathcal{U}}\lambda \in j_{\mathcal{U}}\left(\underset{\sim}{X}\right)$. Assume that $r \in G$ extends both p, q. Then $r^\frown \langle \langle\rangle, \underset{\sim}{A}\rangle^\frown s_\alpha \Vdash j''_{\mathcal{U}}\lambda \in j_{\mathcal{U}}\left(\underset{\sim}{Y}\right)$.

Finally, assume that $p \in G$ forces that a sequence of names $\langle \underset{\sim}{X}_i \colon i < \delta\rangle$, where $\delta < \kappa$, is a partition of $\mathcal{P}_\kappa(\lambda)$. Then $j_{\mathcal{U}}(p)$ forces that $\langle j_{\mathcal{U}}\left(\underset{\sim}{X}_i\right) : i < \delta\rangle$ is a partition of $j_{\mathcal{U}}\left(\mathcal{P}_\kappa(\lambda)\right)$. Hence the set of conditions in $j_{\mathcal{U}}(P_\kappa) \setminus \kappa + 1$ which force, for some $i^* < \delta$, that $j''_{\mathcal{U}}\lambda \in j_{\mathcal{U}}\left(\underset{\sim}{X}_{i^*}\right)$, is forced by $p^\frown 0_{\underset{\sim}{Q}_\kappa}$ to be a $\leq^*$–dense open subset of $j_{\mathcal{U}}(P) \setminus \kappa + 1$ above $\underset{\sim}{s}$. Therefore, for some $\alpha < \lambda^+$, s_α is forced to belong to this dense open set. It follows that–

$$p^\frown 0_{\underset{\sim}{Q}} \Vdash \exists i^* < \delta \;\; s_\alpha \Vdash j''_{\mathcal{U}}\lambda \in j\left(\underset{\sim}{X}_{i^*}\right)$$

and by applying the κ-closure of the direct extension order of $\underset{\sim}{Q}_{j_{\kappa,0}(\kappa)}$, there exists a name $\underset{\sim}{A}$ such that–

$$p \Vdash \exists i^* < \delta \;\; \langle \langle\rangle, \underset{\sim}{A}\rangle^\frown s_\alpha \Vdash j''_{\mathcal{U}}\lambda \in j\left(\underset{\sim}{X}_{i^*}\right)$$

so, finally, by extending p to a condition $q \in G$ which decides the value of i^*, there exists $i^* < \delta$ such that–

$$q^\frown \langle \langle\rangle, \underset{\sim}{A}\rangle^\frown s_\alpha \Vdash j''_{\mathcal{U}}\lambda \in j\left(\underset{\sim}{X}_{i^*}\right)$$

as desired. □

Chapter 3

The Full Support

3.1 Introduction

In this chapter we revisit the Full support iteration of Prikry forcings (the Magidor iteration) which was first introduced by M. Magidor in his celebrated paper [18]. Assuming that κ is a measurable limit of measurables, the Magidor iteration can be used to destroy the measurability of every measurable cardinal $\alpha < \kappa$, while preserving cardinals and the measurability of κ itself. Let P be such an iteration and $G \subseteq P$ generic over the ground model V.

Our first goal would be to characterize the normal measures over κ in $V[G]$. This was extensively studied by Ben-Neria in [2]. For every normal measure $U \in V$ on κ, he assigned a corresponding measure $U^{\times} \in V[G]$ on κ, and showed that the mapping $U \mapsto U^{\times}$ is a bijection between the set of normal measures on κ in V, and the set of normal measures on κ in $V[G]$. This was done under the assumptions that $0^{\P}$ does not exist and the ground model V is the core model. In this chapter, we extend this result, weakening the assumption on the ground model V:

Theorem 3.1.1. *Assume $GCH_{\leq\kappa}$ holds in V. Let $W \in V[G]$ be a normal measure on κ. Then $W = U^{\times}$ for some normal measure $U \in V$ on κ. Moreover, the measures $\langle U^{\times} \colon U \in V \text{ is a normal measure on } \kappa \rangle$ are pairwise distinct.*

The proof relies on some of the methods presented by Ben-Neria in [2]; however, the core-model theoretic aspects of the argument are replaced with the tools developed in [8].

We then proceed and study the structure of $j_W \restriction_V$ for every normal measure $W \in V[G]$ on κ.

Theorem 3.1.2. *Assume $GCH_{\leq\kappa}$ holds in V. Let $W \in V[G]$ be a normal measure on κ. Then $j_W \restriction_V$ is an iterated ultrapower of V by normal measures.*

Moreover, a concrete description of $j_W \restriction_V$ as an iterated ultrapower is given.

Finally, we discuss definability of $j_W \restriction_V$ as a class of V. In general, $j_W \restriction_V$ may not be definable in V (see remark 3.4.25, and, more generally, section 5.2 in [8]). We provide a sufficient condition for definability of $j_W \restriction_V$ as a class of V. By Theorem 3.1.1, given a measurable $\alpha < \kappa$, the measure used in the Prikry forcing at stage α in the iteration P must have the form $U^{\times}_{\alpha} = (U_\alpha)^{\times}$, for some normal measure U_α on α in V. Denote $\vec{\mathcal{U}} = \langle U_\alpha \colon \alpha < \kappa,\ \alpha \text{ is measurable in } V\rangle$. Then:

Theorem 3.1.3. *Assume $GCH_{\leq\kappa}$ holds in V. If $\vec{\mathcal{U}} \in V$ then $j_W \restriction_V$ is a definable class of V.*

We remark that it is not necessarily the case that $\vec{\mathcal{U}} \in V$, even if $j_W \restriction_V$ is a definable class of V (see remark 3.4.25).

This chapter is organized as follows: In the first section we present the forcing and its basic properties. In section 2 we prove theorem 3.1.1. In section 3 we prove theorems 3.1.2 and 3.1.3, and completely describe the Prikry sequences added to measurables of M above κ in H.

3.2 The Forcing

Definition 3.2.1. *An iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \colon \alpha \leq \kappa\ ,\ \beta < \kappa\rangle$ is called a full support (Magidor) iteration of Prikry-type forcings if and only if, for every $\alpha \leq \kappa$ and $p \in P_\alpha$,*

1. *p is a function with domain α such that for every $\beta < \alpha$, $p \restriction \beta \in P_\beta$, $p \restriction \beta \Vdash p(\beta) \in \underset{\sim}{Q}_\beta$ and $\langle \underset{\sim}{Q}_\beta, \underset{\sim}{\leq}_{\underset{\sim}{Q}_\beta}, \underset{\sim}{\leq}^*_{\underset{\sim}{Q}_\beta}\rangle$ is a Prikry-type forcing.*
2. *There exists a finite set $b \subseteq \kappa$ such that for every $\beta \notin b$, $p \restriction_\beta \Vdash p(\beta) \underset{\sim}{\geq}^*_\beta \underset{\sim}{0}_{Q_\beta}$, where $\underset{\sim}{\geq}^*_\beta$ is the direct extension order of $\underset{\sim}{Q}_\beta$.*

Suppose that $p, q \in P_\alpha$. Then $p \geq q$, which means that p extends q, holds if and only if:

1. *For every $\beta \leq \alpha$, $p \restriction \beta \Vdash p(\beta) \geq_\beta q(\beta)$ (where $\geq_\beta$ is the order of Q_β).*
2. *There is a finite subset $b \subseteq \alpha$, such that for every $\beta \in \alpha \setminus b$, $p \restriction \beta \Vdash p(\beta) \geq^*_\beta q(\beta)$ (where $\geq^*_\beta$ is the direct extension order of Q_β).*

If $b = \emptyset$, we say that p is a direct extension of q, and denote it by $p \geq^* q$.

Let $\langle P_\alpha, \underset{\sim}{Q}_\beta \colon \alpha \leq \kappa\ ,\ \beta < \kappa\rangle$ be a full support iteration of Prikry forcings, such that, for every V-measurable cardinal, α, $\underset{\sim}{Q}_\alpha$ is non-trivial, and is forced to be Prikry forcing with a given P_α-name for a normal measure on α (we will prove in lemma 3.3.2 that α remains measurable after forcing with P_α). If α is not measurable in V, $\underset{\sim}{Q}_\alpha$ is the trivial forcing.

We denote $\Delta = \{\alpha < \kappa \colon \alpha \text{ is measurable in } V\}$. For every $\alpha \in \Delta$, let $\underset{\sim}{W}_\alpha$ be the P_α-name for a normal measure on α, which is forced by P_α to be the measure used in the Prikry forcing $\underset{\sim}{Q}_\alpha$. Assume that $p \in P_\kappa$ is a given condition and $\alpha \in \Delta$. We denote by $\underset{\sim}{t}^{\,p}_\alpha$ and $\underset{\sim}{A}^p_\alpha$ the P_α-names such that $p \restriction_\alpha \Vdash p(\alpha) = \langle \underset{\sim}{t}^{\,p}_\alpha, \underset{\sim}{A}^p_\alpha \rangle$. In $V[G]$, we denote by $d \colon \Delta \to \kappa$ the function which maps each former measurable in Δ to the first element in its Prikry sequence. Finally, we adopt the following useful notation, introduced by O. Ben-Neria in [2]: Given a condition $p \in P_\kappa$ and $\alpha < \kappa$, let $p^{-\alpha} \geq^* p$ be the condition p^* which satisfies, for every measurable $\xi > \alpha$,

$$p^* \restriction_\xi \Vdash \underset{\sim}{A}^{p^*}_\xi = \underset{\sim}{A}^p_\xi \setminus (\alpha + 1)$$

The following lemma is standard (see [7] for example):

Lemma 3.2.2. *$P = P_\kappa$ satisfies the Prikry property.*

The main ideas in the proof of the Prikry property of P_κ appear also in the proof of the following Fusion property:

Lemma 3.2.3 (Fusion Lemma)**.** *Let $\delta \leq \kappa$ be a limit ordinal and $p \in P_\delta$. For every $\alpha < \delta$, let $q(\alpha)$ be a P_α-name such that $p \restriction_\alpha \Vdash q(\alpha) \geq^* p \setminus \alpha$. Then there exist $p^* \geq^* p$ such that for every $\alpha < \delta$,*

$$p^* \restriction_\alpha \Vdash (p^* \setminus \alpha)^{-\alpha} \geq^* q(\alpha)$$

Before proving the lemma, let us state an immediate useful corollary of it.

Corollary 3.2.4. *Let $\delta \leq \kappa$ be a limit ordinal and $p \in P_\delta$. For every $\alpha < \delta$, let $e(\alpha)$ be a P_α-name such that–*

$$p \restriction_\alpha \Vdash \text{"}e(\alpha) \textit{ is a dense open subset of } P \setminus \alpha \textit{ above } p \setminus \alpha,$$

$$\textit{with respect to the direct extension order."}$$

Then there exist $p^* \geq^* p$ such that for every $\alpha < \delta$,

$$p^* \restriction_\alpha \Vdash (p^* \setminus \alpha)^{-\alpha} \in e(\alpha)$$

Proof of lemma 3.2.3. Define a sequence $\langle p_\xi \colon \xi > \delta \rangle$ of direct extensions of p, such that for every $\xi < \delta$, $p_\xi \restriction_\xi \Vdash p_\xi \setminus \xi \geq^* q(\xi)$, and, for every $\eta < \xi < \delta$,

1. $p \leq^* p_\eta \leq^* (p_\xi)^{-\eta}$.

2. $p_\eta \restriction_\eta = p_\xi \restriction_\eta$.

Take $p_0 = p$. Assume that $\xi < \kappa$ and $\langle p_\eta \colon \eta < \xi \rangle$ have been defined. Let us define p_ξ. First, set–

$$p_\xi \restriction_\xi = \bigcup_{\xi' < \xi} p_{\xi'} \restriction_{\xi'}$$

We now define a P_ξ-name for a condition $r \in P \setminus \xi$. If ξ is non-measurable, $r(\xi)$ (the value of r at coordinate ξ) is trivial. If it is: Let $\underset{\sim}{t}$ be a P_ξ-name, and, for every $\xi' < \xi$, take P_ξ-names $\underset{\sim}{A}_{\xi'}$ such that $p_\xi \restriction_\xi \Vdash p_{\xi'}(\xi) = \langle \underset{\sim}{t}, \underset{\sim}{A}_{\xi'} \rangle$. Set $r(\xi) = \langle \underset{\sim}{t}, \triangle_{\xi' < \xi} \underset{\sim}{A}_{\xi'} \rangle$. Finally, let $r \setminus (\xi + 1)$ be a direct extension of all the conditions $\langle p_{\xi'} \setminus (\xi + 1) : \xi' < \xi \rangle$ (the direct extension order above ξ is more than ξ-closed). This defines r. Since every pair of direct extensions of $p \setminus \xi$ have a common direct extension, we can pick $p_\xi \setminus \xi$ such that it direct extends both r and $q(\xi)$. Note that $\underset{\sim}{A}_\xi^{p_\xi} \subseteq \triangle_{\xi' < \xi} \underset{\sim}{A}_\xi^{p_{\xi'}}$, and thus, for every $\eta < \xi$, $\underset{\sim}{A}_\xi^{p_\xi} \setminus (\eta + 1) \subseteq \underset{\sim}{A}_\xi^{p_\eta}$. Thus $p_\eta \leq^* (p_\xi)^{-\eta}$.

This finishes the construction. Define $p^* = \bigcup_{\xi < \delta} p_\xi \restriction_\xi$. We claim that p^* is as desired. Let $\alpha < \delta$. Then $p^* \restriction_\alpha = p_\alpha \restriction_\alpha$. Thus, this condition forces that $(p_\alpha \setminus \alpha)^{-\alpha} \in e(\alpha)$. It also forces that $(p^* \setminus \alpha)^{-\alpha}$ direct extends $(p_\alpha \setminus \alpha)^{-\alpha}$, and thus it direct extends $q(\alpha)$, as desired. □

Lemma 3.2.5. *$P = P_\kappa$ preserves cardinals.*

Proof. We prove by induction that for every $\delta \leq \kappa$, P_δ preserves cardinals. This is clear for successor values of δ. By $\mathrm{GCH}_{\leq\kappa}$, this is clear as well if δ is not a limit of measurables. Thus, let us assume that $\delta \leq \kappa$ is a limit of measurables and μ is a cardinal. If $\mu < \delta$, factor $P_\delta = P_{<\mu} * \underset{\sim}{Q}_\mu * P_{>\mu}$. Since the direct extension order of $P_{>\mu}$ is more than μ-closed, it preserves μ; Q_μ preserves μ because it is either trivial or a Prikry forcing; finally, by induction, $P_{<\mu}$ preserves μ. If $\mu = \delta$, then μ is a limit of measurables, each of them is preserved by induction. If $\mu > \delta^+$, μ is preserved since $|P_\delta| = \delta^+$ by $\mathrm{GCH}_{\leq\kappa}$. Thus, it suffices to prove that P_δ preserves δ^+ for every limit of measurables δ. It suffices to prove that P_δ has the $\delta^+ - c.c.$: For any antichain $A \subseteq P_\delta$ of cardinality δ^+, there exists a subset $A' \subseteq A$ of cardinality δ^+, such that the following holds: There exists a finite set $b \subseteq \delta$, and, for every $\alpha \in b$, a P_α-name for a finite increasing sequence $\underset{\sim}{t}_\alpha \in [\alpha]^{<\omega}$, such that–

$$\forall p \in A' \; \forall \beta \in \delta \setminus b \; , p \restriction_\beta \Vdash p(\beta) \geq^* 0_Q$$

and–

$$\forall p \in A' \; \forall \alpha \in b \; \exists \underset{\sim}{A}, p \restriction_\alpha \Vdash p(\alpha) = \langle \underset{\sim}{t}_\alpha, \underset{\sim}{A} \rangle$$

Given these properties, every pair of conditions in A' are compatible, which is a contradiction. $\square$

Lemma 3.2.6. *$P = P_\kappa$ doesn't add fresh subsets of κ, κ^+.*

The above lemma is proved, for example, in [8]. We remark that this proof uses the fact that some normal measure on κ in V extend to a normal measure in $V[G]$, and this is indeed the case (this is well known, and in any case, will be proved in the next section in lemma 3.3.2. The proof will not rely on the current lemma or its consequences).

In [8] it is proved that, if a forcing notion P preserves cardinals and does not add fresh subsets to cardinals in the interval $\left[\kappa, (2^\kappa)^V\right]$, then every κ-complete ultrafilter in the generic extension extends a κ-complete ultrafilter of V. Since we assume $\mathrm{GCH}_{\leq\kappa}$, the following follows:

Corollary 3.2.7. *Let $G \subseteq P_\kappa$ be generic over V, and let $W \in V[G]$ be a κ-complete ultrafilter on κ. Then $W \cap V \in V$.*

We conclude this section by proving a property of $P = P_\kappa$, which will be applied several times throughout this paper.

Lemma 3.2.8. *Let $\delta \leq \kappa$ be an inaccessible cardinal. Let $p \in P_\delta$ and assume that $\underset{\sim}{\alpha}$ is a P_δ-name for an ordinal. Then there exists $p^* \geq^* p$ and a set $A \in V$ with $|A| < \delta$ such that $p^* \Vdash \underset{\sim}{\alpha} \in A$.*

Proof. Denote by D the dense open subset of P_δ which consists of conditions which decide the value of $\underset{\sim}{\alpha}$. We will apply on D the following claim:

Claim 3.2.9. *Let $\delta \leq \kappa$ be a limit ordinal and let $D \subseteq P_\delta$ be a dense open subset of P_δ. Assume that $p \in P_\delta$. Then there exists $p^* \geq^* p$ such that for every $p^* \leq q \in D$,*

$$q \restriction_{\gamma+1}{}^\frown (p^* \setminus (\gamma+1))^{-(\gamma+1)} \in D$$

where γ is the maximal coordinate which satisfies–

$$q \restriction_\gamma \Vdash \text{"}q(\gamma) \text{ is not a direct extension of } p^*(\gamma)\text{"}$$

(and, if such γ does not exist, then $\gamma = 0$).

Proof. Fix a non-measurable $\alpha < \delta$ and $G_\alpha \subseteq P_\alpha$ generic over V such that $p \restriction_\alpha \in G_\alpha$. Given $p \restriction_\alpha \leq q \in G_\alpha$, we define a subset of $P \setminus \alpha$ which is $\leq^*$-dense open above $p \setminus \alpha$:

$$e_q(\alpha) = \{r \in P \setminus \alpha \colon q^\frown r \in D \text{ or } (\forall r' \geq^* r,\ q^\frown r' \notin D)\}$$

Since α is non-measurable, the direct extension order of $P \setminus \alpha$ is more than $|G_\alpha|^+$-distributive. Let $e(\alpha)$ be a P_α-name for the set–

$$e(\alpha) = \bigcap_{q \in G_\alpha} e_q(\alpha)$$

then $p \restriction_\alpha$ forces that $e(\alpha)$ is $\leq^*$-dense open above $p \setminus \alpha$.

Apply lemma 3.2.3. Let $p^* \geq^* p$ be such that, for every non-measurable $\alpha < \delta$,

$$p^* \restriction_\alpha \Vdash (p^* \setminus \alpha)^{-\alpha} \in e(\alpha)$$

Assume now that $p^* \leq q \in D$. Let γ be as in the formulation of the claim. Then $\gamma + 1$ is not measurable, so–

$$p^* \restriction_{\gamma+1} \Vdash (p^* \setminus (\gamma+1))^{-\gamma+1} \in e(\gamma+1)$$

In particular,

$$q \restriction_{\gamma+1} \Vdash (p^* \setminus (\gamma+1))^{-\gamma+1} \in e(\gamma+1)$$

Finally, since there exists a direct extension $r' = q \setminus (\gamma+1) \geq^* p^* \setminus (\gamma+1)$ such that $q \restriction_{\gamma+1} {}^\frown r' \in D$, it follows that $q \restriction_{\gamma+1} {}^\frown (p^* \setminus (\gamma+1))^{-\gamma+1} \in D$, as desired. $\square$

Pick a direct extension $q \geq^* p$, by applying the claim on the set D of conditions deciding the value of $\underset{\sim}{\alpha}$. We will construct below a direct extension $q^* \geq^* q$; After this is done, we will prove that q^* has a direct extension $p^* \geq^* q^*$ as desired in the lemma. Namely, p^* satisfies that for some set of ordinals A with $|A| < \delta$, $p^* \Vdash \underset{\sim}{\alpha} \in A$.

First, let us construct $q^* \geq^* q$. Assume that $\gamma < \delta$, and $q^* \restriction_\gamma$ has been defined. To define $q^*(\gamma)$, we shrink the set $\underset{\sim}{A}^q_\gamma$. We shrink it to a set $A \in W_\gamma$, such that, for every $n < \omega$, exactly one of the following holds: Either for every $s \in [A]^n$, there exists a set of ordinals A_s with $|A_s| < \delta$, such that–

$$\langle t^q_\gamma {}^\frown s, A \setminus \max(s) \rangle^\frown (q \setminus (\gamma+1))^{-\gamma+1} \Vdash \underset{\sim}{\alpha} \in A_s$$

or, there is no such s.

This results in a direct extension $q^* \geq^* q$. It suffices to prove that q^* has a direct extension p^* which belongs to D. Assume otherwise. Let $r \geq q^*$ be a condition in D, which is chosen with the least number of non-direct extensions. Let γ be the maximal coordinate in which a non-direct extension was taken in the extension $r \geq q^*$. Clearly $r \geq q$, and in this extension, as well, γ is the maximal coordinate in which a non-direct extension is taken. Thus, by the choice of q,

$$r \restriction_{\gamma+1} {}^\frown (q \setminus (\gamma+1))^{-\gamma+1} \in D$$

Let $n < \omega$ be such that $r \restriction_\gamma$ forces that $\mathrm{lh}\left(t^r_\gamma\right) = n + \mathrm{lh}\left(t^q_\gamma\right)$. Then $r \restriction_\gamma$ forces that for every $s \in \left[\underset{\sim}{A}^r_\gamma\right]^n$, there exists a set A_s with $|A_s| < \delta$, such that–

$$\langle t^q_\gamma{}^\frown s, \underset{\sim}{A}^r_\gamma \setminus \max(s)\rangle^\frown (q \setminus (\gamma+1))^{-\gamma+1} \Vdash \underset{\sim}{\alpha} \in A_s$$

By taking union on the possible values of the sets A_s as above, there exists a set $A \in V$ with $|A| < \delta$ such that–

$$r \restriction_\gamma{}^\frown \langle \underset{\sim}{t}^q_\gamma, \underset{\sim}{A}^r_\gamma\rangle^\frown (q \setminus (\gamma+1))^{-\gamma+1} \Vdash \underset{\sim}{\alpha} \in A$$

and this contradicts the minimality of the number of non-direct extensions in the choice of $r \geq q^*$. □

Corollary 3.2.10. *Assume that $\delta \leq \kappa$ is inaccessible, $p \in P_\delta$ and let $\underset{\sim}{f}$ be a P_δ-name for a function from δ to the ordinals. Then there exists $p^* \geq^* p$ and a function $F\colon \delta \to [Ord]^{<\delta}$ in V, such that for every $\alpha < \delta$,*

$$(p^*)^{-\alpha} \Vdash \underset{\sim}{f}(\alpha) \in F(\alpha)$$

Proof. For every $\alpha < \delta$, set–

$$e(\alpha) = \{r \in P \setminus \alpha\colon \text{there exists } A \subseteq \mathrm{Ord} \text{ with } |A| < \delta \text{ such that } r \Vdash \underset{\sim}{f}(\alpha) \in A\}$$

by lemma 3.2.8, $e(\alpha)$ is $\leq^*$-dense open. Thus, by Fusion, there exists $p^* \geq^* p$ such that for every $\alpha < \delta$,

$$p^* \restriction_\alpha \Vdash \text{there exists } A_\alpha \subseteq \mathrm{Ord} \text{ with } |A_\alpha| < \delta \text{ such that } (p^* \setminus \alpha)^{-\alpha} \Vdash \underset{\sim}{f}(\alpha) \in A_\alpha$$

Finally, for every $\alpha < \delta$, let $F(\alpha) = \{\beta\colon \exists q \geq p^* \restriction_\alpha,\ r \Vdash \beta \in \underset{\sim}{A}_\alpha\}$. Then $(p^*)^{-\alpha} \Vdash \underset{\sim}{f}(\alpha) \in F(\alpha)$ and $|F(\alpha)| < \delta$, as desired. □

3.3 Normal Measures in the Generic Extension

This section is devoted to the proof of theorem 3.1.1. The same result was first observed by O. Ben-Neria in [2], assuming that V is the core model and there is no inner mode with overlapping extenders. We will reduce the assumptions on V to $\mathrm{GCH}_{\leq\kappa}$.

Throughout this section, we will extensively use arguments and notations introduced in [2]: For every normal measure on κ, $U \in V$, we will define a measure $U^* \in V[G]$ which extends U. It will turn out that U^* is normal if and only if $o(U) = 0$. Let $U^\times$ be the normal measure below U^* in the Rudin-Keisler order. We will prove that every normal measure on κ in $V[G]$ has the form $U^\times$ for some $U \in V$.

We prove theorem 3.1.1 by induction. Thus, we assume in this section that for every $\xi < \kappa$, the measure W_ξ used to singularize ξ, already has the form $U_\xi^\times$ for some normal measure $U_\xi \in V$ on ξ.

Remark 3.3.1. *In [2], as in other applications of the Magidor iteration, it was assumed that the measures $\langle W_\xi \colon \xi \in \Delta \rangle$, which were used to singularize the measurables of Δ, are all derived from normal measures of Mitchell order 0 (in the sense that, for every $\xi \in \Delta$, there exists $U_\xi \in V$ of order 0, such that $W_\xi = U_\xi^*$). We do not assume this in the current paper. Each measure W_ξ has, by induction, the form $U_\xi^\times$ for some normal measure $U_\xi \in V$, but U_ξ does not necessarily has Mitchell order 0.*

We start by extending every normal measure $U \in V$ on κ, to a measure $U^* \in V[G]$. For every P_κ-name $\underset{\sim}{A}$ for a subset of κ, $(\underset{\sim}{A})_G \in U^*$ if and only if, for some $p \in G$,

$$\{\xi < \kappa \colon p^{-\xi} \Vdash \check{\xi} \in \underset{\sim}{A}\} \in U$$

or simply $(j_U(p))^{-\kappa} \Vdash \check{\kappa} \in j_U(\underset{\sim}{A})$ in M_U.

Lemma 3.3.2. *U^* is a measure on κ in $V[G]$ which extends U. Moreover, U^* is normal if and only if U has Mitchell order 0 in V.*

Proof. It's not hard to verify that U^* is a filter which extends U. Let us prove that it is a κ-complete ultrafilter. Assume that $\langle \underset{\sim}{A}_\xi \colon \xi < \delta \rangle$ is forced by a condition $p \in G$ to be a partition of κ, for some $\delta < \kappa$. Assume that q is an arbitrary condition above p. For every $\alpha \in (\delta, \kappa)$, consider the P_α-name for the following set $e(\alpha)$, which is forced by $q \restriction_\alpha$ to be $\leq^*$-dense open above $q \setminus \alpha$,

$$e(\alpha) = \{r \geq^* q \setminus \alpha \colon \exists \xi^* < \delta,\ r \Vdash \check{\alpha} \in \underset{\sim}{A}_{\xi^*}\}$$

by lemma 3.2.3, there exists $p^* \in G$ above p, such that for every $\alpha \in (\delta, \kappa)$,

$$p^* \restriction_\alpha \Vdash \exists \xi^* < \delta,\ (p^* \setminus \alpha)^{-\alpha} \Vdash \check{\alpha} \in \underset{\sim}{A}_\xi$$

and thus–

$$p^* \Vdash \exists \xi^* < \delta,\ (j_U(p^*))^{-\kappa} \setminus \kappa \Vdash \check{\kappa} \in j_U(\underset{\sim}{A}_{\xi^*})$$

by extending p^* to a stronger condition in G, we can assume that p^* decides the value of ξ^*, and so, for some $\xi^* < \kappa$,

$$(j_U(p^*))^{-\kappa} \Vdash \check{\kappa} \in j_U(\underset{\sim}{A}_{\xi^*})$$

as desired.

Let us assume that U has Mitchell order 0. Let $\underset{\sim}{f}$ be a P_κ-name for a regressive function, as forced by some $p \in G$. We use a similar argument as before, but now $e(\alpha)$ is defined for every non-measurable α, to be the name for the following set, which is forced by any extension of $p \restriction_\alpha$ to be $\leq^*$-dense open above $p \setminus \alpha$:

$$e(\alpha) = \{r \in P \setminus \alpha \colon \exists \xi^* < \alpha,\ r \Vdash \underset{\sim}{f}(\alpha) = \xi^*\}$$

where we used the fact that α is not measurable, and thus $\langle P \setminus \alpha, \leq^* \rangle$ is more than α-closed. Thus, there exists $p^* \in G$ such that-

$$p^* \Vdash \exists \xi^* < \kappa,\ (j_U(p^*) \setminus \kappa)^{-\kappa} \Vdash j_U(\underset{\sim}{f})(\kappa) = \xi^*$$

By extending p^* to a condition in G, we can assume that p^* decides the value of ξ^*. Thus, $\{\xi < \kappa \colon f(\xi) = \xi^*\} \in U^*$, as desired.

Finally, assume that U^* is normal. Let $j_{U^*} \colon V[G] \to M[H]$ be the ultrapower embedding. Note that κ is not measurable in M, since, else, κ would have been singular in $M[H]$, and therefore also in $V[G]$. Thus,

$$\kappa \in j_{U^*}(\{\xi < \kappa \colon \xi \text{ is not measurable in } V\})$$

and thus $U = U^* \cap V$ concentrates on non-measurables. $\square$

Let us define the measure $U^\times \in V[G]$.

Definition 3.3.3. *Assume that $U \in V$ is a normal measure on κ. If U has Mitchell order 0, define $U^\times = U^*$. Assume otherwise. Let $d \colon \Delta \to \kappa$ be the function which maps every measurable cardinal of V to the first element in its Prikry sequence in $V[G]$. Set–*

$$U^\times = d_*(U^*) = \{A \subseteq \kappa \colon d^{-1}[A] \in U^*\}$$

We will prove that whenever U^* is non-normal, namely, $\Delta \in U^*$, d projects U^* to the normal measure below it in the Rudin-Keisler order; this projected measure is $U^\times$ defined above.

Lemma 3.3.4. *Let U be a normal measure on κ in V. Then $U^{\times}$ is a normal measure on κ in $V[G]$.*

Proof. We can assume that U has Mitchell order > 0. It suffices to prove that $[d]_{U^*} = \kappa$.

First, note that for every $x < \kappa$, $d^{-1}\{x\}$ is finite. Indeed, given an arbitrary condition $p \in P_\kappa$, let $b \subseteq \kappa$ be the finite set such for every $\xi \in \kappa \setminus b$, $p \restriction_\xi \geq^* 0_{Q_\xi}$. For every such ξ, let $p^* \geq^* p$ be such that x is removed from every measure one set. Then p^* forces that $d^{-1}\{x\}$ is finite, and since p was arbitrary, this indeed holds in $V[G]$.

This shows that $[d]_{W^*} \geq \kappa$. Assume that $f \in V[G]$ is a function in $V[G]$ such that, for every $\xi \in \Delta$, $f(\xi) < d(\xi)$. Let p be a condition which forces this. Assume that $q \geq p$ is arbitrary, and let ξ_0 be an ordinal which such that for every $\xi > \xi_0$, $q \restriction_\xi \Vdash q(\xi) \geq^* 0_{\underset{\sim}{Q}_\xi}$. For every $\xi \in \Delta$ above ξ_0, we describe a name for a subset of $P \setminus \xi$ which is forced by $q \restriction_\xi$ to be $\leq^*$ dense open subset of $P \setminus \xi$ above $q \setminus \xi$,

$$e(\xi) = \{r \geq q \setminus \xi \colon \text{there exists } \gamma < \xi \text{ such that } r \setminus \xi \Vdash \underset{\sim}{f}(\xi) = \gamma\}$$

The density follows since every name for an ordinal below the first element for a Prikry sequence can be decided by a direct extension.

By fusion, there exists $p^* \in G$ above p such that–

$$p^* \Vdash \exists \gamma < \kappa,\ (j_U(p) \setminus \kappa)^{-\kappa} \Vdash j_U(\underset{\sim}{f})(\kappa) = \check{\gamma}$$

and by extending p^* to a condition in G, we can assume that it decides the value of $\gamma < \kappa$. So $(j_U(p^*))^{-\kappa} \Vdash j_U(\underset{\sim}{f})(\kappa) = \check{\gamma}$, and thus, in $V[G]$, $[f]_{U^*} = \gamma < \kappa$, as desired. □

Claim 3.3.5. *Let $U \in V$ be a normal measure on κ. The following are equivalent:*

1. *U has Mitchell order 0 in V.*
2. *$U^{\times} = U^*$.*
3. *$d''\Delta \notin U^{\times}$.*

Proof. Clearly 1 implies 2 by the definition of $U^{\times}$.

Assume 2. If $d''\Delta \in U^{\times}$ then $d''\Delta \in U^*$, and thus, there exists $p \in G$ such that–

$$(j_U(p))^{-\kappa} \Vdash \check{\kappa} \in j_U(\underset{\sim}{d}''\Delta)$$

but this cannot happen, since $(j_U(p))^{-\kappa}$ forces that κ does not appear as an element in any of the Prikry sequences.

Finally, if U has Mitchell order higher than 0 in V, then $\Delta \in U$ and thus $\Delta \in U^*$. Therefore, $d''\Delta \in U^\times$. $\square$

Lemma 3.3.6. *Let U be a normal measure on κ in V with $o(U) > 0$. Let $j_{U^*}: V[G] \to M[H]$ be the ultrapower embedding of U^*. Then $[Id]_{U^*}$ is measurable in M, and κ appears as a first element in its Prikry sequence in $M[H]$. $[Id]_{U*}$ is maximal with this property, namely, for every measurable above $[Id]_{U^*}$, κ does not appear in its Prikry sequence. Furthermore, for every $\mu > [Id]_{U^*}$ measurable in M, $d(\mu) > [Id]_{U^*}$.*

Proof. Since $\Delta \in U \subseteq U^*$, $[Id]_{U^*}$ is measurable in M. But–

$$\kappa = [d]_{U^*} = j_{U^*}(d)([Id]_{U^*})$$

so κ appears first in the Prikry sequence of $[Id]_{U^*}$ in $M[H]$.

Finally, fix any condition $p \in G$. Then–

$$(j_U(p))^{-\kappa} \Vdash \text{for every } \mu \in j_U(\Delta) \setminus (\kappa+1),\ j_U(\utilde{d})(\mu) > \kappa$$

In particular, $\{\xi < \kappa\colon \text{for every } \mu \in \Delta \setminus (\xi+1),\ d(\mu) > \xi\} \in U^*$. Thus, for every measurable $\mu > [Id]_{U^*}$, $d(\mu) > [Id]_{U^*}$. $\square$

Let us assume now that W is an arbitrary normal measure on κ in $V[G]$. Our goal will be to prove that $W = U^\times$ for some normal measure $U \in V$. Denote by $j_W: V[G] \to M[H]$ the ultrapower embedding of W over $V[G]$. We start with the following observation:

Claim 3.3.7. *Let W be a normal measure on κ in $V[G]$. Then–*

$$\kappa \setminus \bigcup_{\alpha\in\Delta} (d(\alpha), \alpha] \in W$$

Proof. [1] Assume otherwise. Then $X = \bigcup_{\alpha\in\Delta} (d(\alpha), \alpha] \in W$. We argue that there exists a regressive function $f\colon X \to \kappa$ which is not constant modulo W (this is a contradiction, since $W \in V[G]$ is normal, and hence all the sets in W are stationary in κ). Indeed, for every $\eta \in X$, let $\alpha_\eta \in \Delta$ be the first α such that $\eta \in (d(\alpha), \alpha]$. Then define $f(\eta) = d(\alpha_\eta)$. f is not constant modulo W since otherwise there exists $\xi < \kappa$ with $d^{-1}\{\xi\}$ infinite. $\square$

[1] The proof presented here was offered by Omer Ben-Neria, and is a major simplification of the original argument.

Remark 3.3.8. *$W \cap V$ is a normal measure in V of Mitchell order 0. Indeed, by corollay 3.2.7, $W \cap V \in V$. Clearly $W \cap V$ is normal in V. Finally, note that $\Delta \notin W \cap V$, namely $\kappa \notin j_W(\Delta)$. Otherwise, κ was measurable in M, and thus singular in $M[H] \subseteq V[G]$. But κ is regular in $V[G]$, a contradiction.*

Let us assume, by induction, that for every measurable $\mu < \kappa$, the normal measures on μ in V^{P_μ} have the form $U^\times$ for some normal measure U on μ in V. From the previous remark, we can assume also that every such $U^\times$ concentrates on non-measurables of V below μ.

Definition 3.3.9. *Let $W \in V[G]$ be a normal measure on κ. We now define a normal measure $W^* \in V[G]$ on κ. If $d''\Delta \notin W$, take $W^* = W$. Assume otherwise. For every $\delta < \kappa$, the set $d^{-1}\{\delta\}$ is finite (see the proof of lemma 3.3.4). Define a set $\Delta^* \subseteq \Delta$,*

$$\Delta^* = \{\xi \in \Delta \colon \xi = \max d^{-1}\{d(\xi)\}\}$$

Δ^* is an unbounded subset of Δ, on which d is injective. Let–

$$W^* = \{X \subseteq \kappa \colon d''(X \cap \Delta^*) \in W\}$$

W^* is a non-trivial, κ-complete ultrafilter on κ.

Let us review some of the properties of W^* in the case where $d''\Delta \in W$. Clearly $\Delta, \Delta^* \in W^*$. d is a Rudin-Keisler projection of W^* onto W, and is injective on $\Delta^* \in W^*$. Therefore $W \equiv_{RK} W^*$, and in particular $j_W = j_{W^*}$, namely W, W^* have the same ultrapower embedding from $V[G]$ to $M[H]$. In $M[H]$, $\kappa = [d]_{W^*} = j_{W^*}(d)([Id]_{W^*})$, namely κ is the first element in the Prikry sequence of $[Id]_{W^*}$. Finally, $\Delta^* \in W^*$, and thus–

$$[Id]_{W^*} = \max j_{W^*}(d)^{-1}\{\kappa\}$$

so κ does not appear as first element in the Prikry sequence of any measurable above $[Id]_{W^*}$.

Lemma 3.3.10. *$W^* \cap V \in V$ is a normal measure on κ in V.*

Proof. By corollary 3.2.7, $W^* \cap V \in V$. If $d''\Delta \notin W$, then $W^* = W$ is normal, and so is $W^* \cap V$. Let us assume that $d''\Delta \in W$. Assume that $f \in V$ and $\{\xi < \kappa \colon f(\xi) < \xi\} \in W^* \cap V$. Denote this set by A and assume that $A \subseteq \Delta$ (else, intersect).

For every $p \in P_\kappa$, there exists a direct extension $p^* \geq^* p$ and a finite subset $b \subseteq \kappa$ such that, for every $\xi \in A \setminus b$,

$$p^* \restriction_\xi \Vdash \underset{\sim}{A}^{p^*}_\xi \subseteq \xi \setminus (f(\xi)+1) \text{ and } t^{p^*}_\xi = \langle\rangle$$

thus, there exists such $b \subseteq \kappa$ and $p^* \in G$. Then p^* forces that for every $\xi \in A \setminus b$, $f(\xi) < d(\xi)$. But $A \in W^*$, and thus $A \setminus b \in W^*$, so, in $M[H]$, $[f]_{W^*} < [d]_{W^*} = d([Id]_{W^*}) = \kappa$. Therefore, there exists $\beta < \kappa$ such that–

$$\{\xi < \kappa \colon f(\xi) = \beta\} \in W^*$$

but this set belongs to V (since $f \in V$), and thus–

$$\{\xi < \kappa \colon f(\xi) = \beta\} \in W^* \cap V$$

as desired. □

Remark 3.3.11. *Given a normal measure on κ, $W \in V[G]$, we abuse the notation and denote by d the function $j_W(d) \colon j_W(\Delta) \to \kappa$. Similarly, given a normal measure $U \in V$ on κ, we use $\underset{\sim}{d}$ to denote the $j_U(P)$-name $j_U(\underset{\sim}{d})$.*

Lemma 3.3.12. *Let $p \in G$ be a condition. Then $(j_W(p))^{-[Id]_{W^*}} \in H$. In particular, if $d''\Delta \notin W$, Then $j_W(p)^{-\kappa} \in H$.*

Proof. $j_W(p) \in H$ since $p \in G$. In order to prove that $(j_W(p))^{-[Id]_{W^*}} \in H$, it suffices to prove that ordinals $\leq [Id]_{W^*}$ do not appear in Prikry sequences of measurables above $[Id]_{W^*}$ in $M[H]$.

Clearly, for every $\mu > [Id]_{W^*}$, $d(\mu) \geq \kappa$. Otherwise, there exist $\alpha < \kappa$ and $A \in W^*$ such that for every $\xi \in A$, there is some $\mu(\xi) > \xi$ with $d(\mu(\xi)) = \alpha$. In particular, $d^{-1}\{\alpha\}$ is infinite, a contradiction.

Let us argue now that for every $\mu > [Id]_{W^*}$, $d(\mu) > \kappa$. If $d''\Delta \notin W$ this is clear, since κ does not belong to the image of d in $M[H]$. Thus, let us take care of the case where $d''\Delta \in W$. In this case, recall that in $M[H]$, $[Id]_{W^*} = \max d^{-1}\{\kappa\}$. Thus, for every $\mu > [Id]_{W^*}$, $d(\mu) \neq \kappa$.

Finally, let us argue that for every $\mu > [Id]_{W^*}$, $d(\mu) > [Id]_{W^*}$. It suffices to prove that for every such μ, $d(\mu) \notin (\kappa, [Id]_{W^*}]$. If $d''\Delta \notin W$ this is clear, since in this case $W^* = W$ and $[Id]_{W^*} = \kappa$. Let us assume that $d''\Delta \in W$. We claim that in $V[G]$, there exists a finite set $b \subseteq \kappa$ such that for every measurable $\mu > \sup(b)$,

$$d(\mu) \notin \bigcup_{\xi \in \Delta \cap \mu} (d(\xi), \xi]$$

We prove this by a density argument. Fix a condition $p \in P_\kappa$. Let $b \subseteq \kappa$ be the set of coordinates such that for every $\mu > \sup(b)$, $p \restriction_\mu \Vdash p(\mu) \geq^* 0$. We extend p to $p^* \geq^* p$ such that, for every measurable $\mu > \sup(b)$,

$$p^* \restriction_\mu \Vdash \underset{\sim}{d}(\mu) \notin \bigcup_{\xi \in \Delta \cap \mu} (d(\xi), \xi]$$

this is possible since, by the induction hypothesis, the weakest condition in P_μ forces that–

$$\bigcup_{\xi \in \Delta \cap \mu} (d(\xi), \xi]$$

does not belong to any normal measure in V^{P_μ}. Pick such $p^* \in G$. Then in $V[G]$, for every $\mu > \sup(b)$,

$$d(\mu) \notin \bigcup_{\xi \in \Delta \cap \mu} (d(\xi), \xi]$$

This is true for every $\mu \in \Delta \setminus \sup(b) \in W^*$. Thus, in $M[H]$, for every $\mu > [Id]_{W^*}$, $d(\mu) \notin (\kappa, [Id]_{W^*}]$. □

Proof of Theorem 3.1.1. Let $W \in V[G]$ be a normal measure on κ. Let $U = W^* \cap V$. Let $k \colon M_U \to M$ be the embedding which satisfies, for every $f \in V$,

$$k\left([f]_U\right) = [f]_{W^*}$$

It's not hard to verify that k is elementary and $j_W \restriction_V = j_{W^*} \restriction_V = k \circ j_U$. Moreover, $\operatorname{crit}(k) > \kappa$ if and only if $d''\Delta \notin W$: Indeed, if $d''\Delta \notin W$ then $W^* = W$ is normal and thus $k(\kappa) = \kappa$, and if $d''\Delta \in W$ then $\kappa = [d]_{W^*} < [Id]_{W^*} = k\left([Id]_U\right) = k(\kappa)$.
Let us argue now that $W = U^\times$.

Assume first that $d''\Delta \notin W$. Then Definition 3.3.9, $W^* = W$, and by Remark 3.3.8, U has Mitchell order 0. We argue that $W = U^\times = U^*$. It suffices to prove that $U^* \subseteq W$. Let $X \in U^*$, and assume that $\underset{\sim}{X} \in V$ is a P_κ-name such that $(\underset{\sim}{X})_G = X$. Then for some $p \in G$,

$$\left(j_U(p)\right)^{-\kappa} \Vdash \check{\kappa} \in j_U\left(\underset{\sim}{X}\right)$$

By applying $k \colon M_U \to M$,

$$\left(j_W(p)\right)^{-\kappa} \Vdash \check{\kappa} \in j_W\left(\underset{\sim}{X}\right)$$

where we used that fact that $k(\kappa) = \kappa$. Since $p \in G$ and $d''\Delta \notin W$, $\left(j_W(p)\right)^{-\kappa} \in H$, and thus, in $M[H]$, $\kappa \in j_W(X)$, as desired.

Assume now that $d''\Delta \in W$. Then $\Delta \in W^*$ and thus $o(U) > 0$. In this case, $k(\kappa) = [Id]_{W^*} > \kappa$. Let us prove that $W = U^\times$. Since both are ultrafilters in $V[G]$, it suffices to prove that $U^\times \subseteq W$. Assume that $X \in U^\times$, and let $\underset{\sim}{X} \in V$ be such that $(\underset{\sim}{X})_G = X$. Let $p \in G$ be a condition such that–

$$(j_U(p))^{-\kappa} \Vdash \check{\kappa} \in j_U\left(\underset{\sim}{d}^{-1}\underset{\sim}{X}\right)$$

By applying $k\colon M_U \to M$,

$$(j_W(p))^{-[Id]_{W^*}} \Vdash \underset{\sim}{d}\left([Id]_{W^*}\right) \in j_W\left(\underset{\sim}{X}\right)$$

but $(j_W(p))^{-[Id]_{W^*}} \in H$ by lemma 3.3.12, and thus, in $M[H]$,

$$\kappa = d\left([Id]_{W^*}\right) \in \left(j_W\left(\underset{\sim}{X}\right)\right)_H = j_W(X)$$

so $X \in W$, as desired.

Finally, assume that $U \neq U'$ are normal measures in V. If both have Mitchell order 0, then $U^* \neq U'^*$ and thus $U^\times \neq U'^\times$. If exactly one of them, say U, has Mitchell order 0, then $d''\Delta \in U'^\times \setminus U^\times$. Thus, let us consider the case where both have Mitchell order higher than 0. Let $A \in U$, $B \in U'$ be disjoint sets. In $V[G]$, let $A^* = A \cap \Delta^* \in U^*$, $B^* = B \cap \Delta^* \in U'^*$. Then $d''A^* \in U^\times$, $d''B^* \in U'^\times$, and $d''A^* \cap d''B^* = \emptyset$ since d is injective on Δ^*. Thus $U^\times \neq U'^\times$. □

The embedding $k\colon M_U \to M$ from the above proof will be used in the next sections to analyze the structure of $j_W\restriction_V$. For now, let us note that $\text{crit}(k) = \kappa$ if and only if $d''\Delta \notin W$.

3.4 The Structure of $j_W\restriction_V$

Given a normal measure $W \in V[G]$ on κ, let $j_W\colon V[G] \to M[H]$ be the ultrapower embedding, and let $U \in V$ be a normal measure on κ such that $W = U^\times$. Our main goal in this section will be to factor $j_W\restriction_V$ to an iterated ultrapower of V.

We divide this section to several subsections. In the first subsection, we isolate a natural number $m < \omega$ and a sequence $U^0 \lhd U^1 \lhd \ldots \lhd U^m = U$ of measures on κ in V. In the second subsection, we describe in detail the structure of $j_W\restriction_V$ and sketch the main steps in the proof. We will also demonstrate the structure of $j_W\restriction_V$ in several simple cases. In the third subsection, we develop a generalization of the Fusion lemma. This generalization will be applied in the fourth subsection, where we complete the proof of theorem 3.1.2, provide a sufficient condition for the definability of

$j_W \restriction_V$ in V, and describe the Prikry sequences added by H for measurables of M above κ. For instance, we will prove that each measure U^j, for $0 \le j < m$, is iterated in $j_W \restriction_V$ ω-many times, producing Prikry sequences for one of the measurables in the finite set $d^{-1}\{\kappa\}$

The value of $m < \omega$ and the exact measures participating in the sequence $U^0 \lhd U^1 \lhd \ldots \lhd U^m = U$ depend on W and on the measures in the sequence $\langle W_\xi \colon \xi \in \Delta\rangle \in V[G]$, namely the measures used in G to singularize the measurables of Δ. For every $\xi \in \Delta$, denote by $U_\xi \in V$ the measure on ξ such that $W_\xi = U_\xi^\times$. By induction, for every $\xi \in \Delta$ there exists a natural number m_ξ and a sequence $U_\xi^0 \lhd \ldots \lhd U_\xi^{m_\xi - 1} \lhd U_\xi^{m_\xi} = U_\xi$ of normal measures on ξ in V. The identity of the measures $\langle U_\xi^i \colon \xi \in \Delta,\ j \le m_\xi\rangle$ determines the measures participating in the iteration of $j_W \restriction_V$, and whether or not this iteration is definable in V.

3.4.1 The System $U^0 \lhd U^1 \lhd \ldots \lhd U^m$ Associated with W

Denote $m = m(W) = \left|d^{-1}\{\kappa\}\right|$ as computed in $M[H]$. Namely, $m(W) < \omega$ is the number of occurrences of κ as a first element in Prikry sequences added to measurables in M. Possibly $m(W) = 0$, in the case where $d''\Delta \notin W$. Define, for every $i \ge 1$, the set $\Delta_i \subseteq \Delta$:

$$\begin{aligned}\Delta_i =& \{\xi \in \Delta \colon \left|\xi \cap d^{-1}\{d(\xi)\}\right| = i - 1\} = \\ & \{\xi \in \Delta \colon \xi \text{ is the } i\text{-th element in } d^{-1}\{d(\xi)\}\}\end{aligned}$$

For $i = 0$, let $\Delta_0 = \kappa \setminus \Delta$, the set of non-measurables below κ. We state some straightforward properties:

Claim 3.4.1.

1. *$\{\xi < \kappa \colon \xi$ appears as first element in m Prikry sequences below $\kappa\} \in W$.*

2. *For all but finitely many $\xi \in \Delta$, if $m(W_\xi) = i - 1$ for some $1 \le i < \omega$, then ξ is the i-th element in $d^{-1}\{d(\xi)\}$.*

3. *$d''\Delta_1 \supseteq d''\Delta_2 \supseteq \ldots \supseteq d''\Delta_n \supseteq \ldots$ ($n < \omega$).*

4. *m is the maximal index such that $d''\Delta_m \in W$.*

Note that d is injective on each of the sets Δ_i. Let us define, for every $1 \le i \le m$, a measure W^i as follows:

$$W^i = \{X \subseteq \kappa \colon d''(X \cap \Delta_i) \in W\}$$

In particular, W^m is the measure W^* defined in the previous section. Since d is injective on each set Δ_i,

$$W \equiv_{RK} W^1 \equiv_{RK} W^2 \equiv_{RK} \ldots \equiv_{RK} W^m = W^*$$

For every $1 \leq i \neq j \leq m$, let $\pi_{i,j} \colon \Delta_i \to \Delta_j$ be the function which maps each $\xi \in \Delta_i$ to the j-th element in $d^{-1}(d\,(\xi))$ (which typically exists. if not, set $\pi_{i,j}(\xi) = 0$). Then $\pi_{i,j}$, which projects W^i onto W^j, is injective on the set–

$$\{\xi \in \Delta_i \colon \left|d^{-1}\left(d(\xi)\right)\right| = m\} \in W^i$$

Finally, denote, for every $1 \leq i \leq m$, $U^i = W^i \cap V \in V$, and note that $U = U^m$.
For sake of completeness, let us denote $W^0 = W$ and $U^0 = W \cap V$. By remark 3.3.8, U^0 concentrates on $\Delta_0 = \kappa \setminus \Delta$. We begin by studying the properties of U^0.

Lemma 3.4.2. *$U^0 \trianglelefteq U$ is a normal measure of Mitchell order 0 in V. $U^0 = U$ if and only if U already has Mitchell order 0 in V. Finally, if U has Mitchell order above 0 in V, then $U^0 = \{A \subseteq \kappa \colon \kappa \in k\,(A)\} \cap M_U$.*

We will need the following claim:

Claim 3.4.3. *Let $U \in V$ be a measure on κ. Then $M_U\,[G]$ and $V\,[G]$ have the same subsets of κ.*

Proof. First let us assume that U concentrates on non-measurables. We will then adjust the proof to the other case. Assume that $\underset{\sim}{A} \in V$ is a P-name for a subset of κ. For every non-measurable $\alpha < \kappa$, let $e(\alpha)$ be the $\leq^*$-dense open subset of $P \setminus \alpha$ which decides the value of $\underset{\sim}{A} \cap \alpha$ over V^{P_α}. By lemma 3.2.3, there exists $p \in G$ such that for every non-measurable $\alpha < \kappa$,

$$p \restriction_\alpha \Vdash (p \setminus \alpha)^{-\alpha} \in e(\alpha)$$

For every such α, let $\underset{\sim}{A}_\alpha \in V_\kappa$ be a P_α-name such that–

$$p \restriction_\alpha \Vdash p \setminus \alpha \Vdash \underset{\sim}{A} \cap \alpha = \underset{\sim}{A}_\alpha$$

The sequence $\langle \underset{\sim}{A}_\alpha \colon \alpha < \kappa \rangle$ belongs to M_U. Thus, $A = (\underset{\sim}{A})_G \in M_U\,[G]$, since–

$$A = \bigcup_{\alpha < \kappa} (\underset{\sim}{A}_\alpha)_{G_\alpha}$$

We now adjust the proof for the case where $\Delta \in U$. We apply Fusion as before. For every $\alpha \in \Delta$, let–

$$e(\alpha) = \{r \in P \setminus \alpha \colon \exists B \subseteq \alpha, r \Vdash \underset{\sim}{A} \cap \underset{\sim}{d}(\alpha) = B \cap \underset{\sim}{d}(\alpha)\}$$

Before proving that $e(\alpha)$ is indeed $\leq^*$-dense open, let us argue that this suffices. By Fusion, there exists $p \in G$, and, for every $\alpha \in \Delta$, a P_α-name $\underset{\sim}{B}_\alpha$ for a subset of α, such that for each such α,

$$p \restriction_\alpha \Vdash (p \setminus \alpha)^{-\alpha} \Vdash \underset{\sim}{A} \cap \underset{\sim}{d}(\alpha) = \underset{\sim}{B}_\alpha \cap \underset{\sim}{d}(\alpha)$$

By closure under κ-sequences, the sequence $\langle \underset{\sim}{B}_\alpha \colon \alpha \in \Delta \rangle$ belongs to M_U. Therefore, in $M_U[G]$, A can be constructed as follows:

$$A = \bigcup_{\alpha \in \Delta} \left((\underset{\sim}{B}_\alpha)_{G_\alpha} \cap d(\alpha) \right)$$

Let us prove now that $e(\alpha)$ is $\leq^*$-dense open. Pick $r \in P \setminus \alpha$. For every $\nu \in \underset{\sim}{A}^r_\alpha$, let $X_\nu \in W_\alpha = U^\times_\alpha$, $B_\nu \subseteq \nu$ and $s_\nu \geq^* r \setminus (\nu + 1)$ be such that–

$$\langle \underset{\sim}{t}^r_\alpha {}^\frown \langle \nu \rangle, X_\nu \rangle^\frown s_\nu \Vdash \underset{\sim}{A} \cap \nu = B_\nu$$

This can be done since the direct extension order of $P \setminus \alpha$ is more than ν-closed. Now let $B = [\nu \mapsto B_\nu]_{W_\alpha}$. Pick $X \in W_\alpha$ such that for every $\nu \in X$, $B \cap \nu = B_\nu$.

Now direct extend r as follows: shrink $\underset{\sim}{A}^r_\alpha$ such that it is contained in $X \cap (\triangle_{\nu<\alpha} X_\nu)$. Then, direct extend $r \setminus (\alpha + 1)$ to be $s_{d(\alpha)}$. Let $r^* \geq^* r$ be the condition obtained this way. Then $r^* \in e(\alpha)$ and this is witnessed by the set $B \subseteq \alpha$. □

Proof of Lemma 3.4.2. If U has Mitchell order 0 in V, then $W = U^\times = U^*$ and thus $U^0 = W \cap V = U$. Let us assume that U has Mitchell order higher than 0, namely $\Delta \in U$.

We provide a definition of U^0 which is different from the definition $U^0 = W \cap V$ as in the statement of the lemma. From the definition we provide, it will be simple to see that $U^0 \in M_U$. After that, we will prove that indeed $U^0 = W \cap V$.

In $V[G]$, define for every $\alpha \in \Delta$, $U^0_\alpha = W_\alpha \cap V \in V$. In V, let $\underset{\sim}{U}^0 = j_U\left(\langle \underset{\sim}{U}^0_\alpha \colon \alpha \in \Delta \rangle\right)(\kappa)$. This is a $j_U(P) \restriction_\kappa = P$-name for a normal measure of Mitchell order 0 which belongs to M_U. Let $U^0 = \left(\underset{\sim}{U}^0\right)_G \in M_U$. Then $U^0 \in V$ is a normal measure on κ of Mitchell order 0. Since $U^0 \lhd U$, it suffices to prove that $U^0 = W \cap V$. Assume that $A \in U^0$ holds, and consider this as a statement in $M_U[G]$. For some $p \in G$,

$$p \Vdash \check{A} \in j_U\left(\langle \underset{\sim}{U}^0_\alpha \colon \alpha \in \Delta \rangle\right)(\kappa)$$

Let $\alpha \mapsto A(\alpha)$ be a function in V which represents A in M_U. Then we can assume that for every $\alpha \in \Delta$,

$$p \restriction_\alpha \Vdash \check{A}(\alpha) \in \underset{\sim}{U}^0_\alpha \subseteq U^\times_\alpha$$

By lemma 3.2.3, there exists $p^* \geq^* p$ such that, for all but finitely $\alpha \in \Delta$,

$$(p^*)^{-\alpha} \Vdash \underset{\sim}{d}(\alpha) \in \check{A}(\alpha)$$

where $d(\alpha)$ is the first element in the Prikry sequence of α. Thus,

$$(j_U(p^*))^{-\kappa} \Vdash \check{\kappa} \in j_U\left(\underset{\sim}{d}^{-1}\left(\check{A}\right)\right)$$

and thus $d^{-1}(A) \in U^*$ in $V[G]$. Therefore,

$$d''\left(d^{-1}A\right) \in U^\times = W$$

so $A \in W$, as desired.

Finally, let us assume that $\Delta \in U$ and argue that $U^0 = \{A \subseteq \kappa \colon \kappa \in k(A)\} \cap M_U$. Since both are ultrafilters in M_U, it suffices to prove that $U^0 \subseteq \{A \subseteq \kappa \colon \kappa \in k(A)\} \cap M_U$.

Let $A \in U^0$ be a set, and assume that $\xi \mapsto A(\xi)$ is a function in V such that $[\xi \mapsto A(\xi)]_U = A$. Assume that $p \in G$ forces that $A \in \underset{\sim}{U}^0$. We can assume that for every $\xi < \kappa$, $p \restriction_\xi \Vdash A(\xi) \in U^0_\xi$, and in particular, $p \restriction_\xi \Vdash A(\xi) \in U^\times_\xi$.

Given any extension $q \geq p$ in P_κ, there exists $p^* \geq^* p$ and a finite subset $b \subseteq \kappa$ such that, for every $\xi \in \Delta \setminus b$,

$$p^* \restriction_\xi \Vdash \underset{\sim}{A}^{p^*}_\xi \subseteq A(\xi) \text{ and } t^{p^*}_\xi = \langle\rangle$$

and thus, there exists such $p^* \in G$. Since $\Delta \setminus b \in W^*$ and $j_{W^*}(p) \in H$, it follows that, in $M[H]$,

$$\kappa = d([Id]_{W^*}) \in [\xi \mapsto A_\xi]_{W^*} = k(A)$$

as desired. □

Lemma 3.4.4. *For every $1 \leq i \leq m$, U^i is normal and has Mitchell order higher than 0. Furthermore,*

$$U^0 = U^0 \lhd U^1 \lhd U^2 \lhd \ldots \lhd U^m = U$$

Proof. The proof that each U^i is normal is identical to 3.3.10, and essentially follows from the fact that d projects each W^i onto W.

For $i \geq 1$, each U^i has Mitchell order above 0: otherwise, $\kappa \setminus \Delta \in U^i \subseteq W^i$, and this contradicts the fact that $\Delta_i \in W^i$ is disjoint from $\kappa \setminus \Delta$.

Let us prove that for every $1 \leq i < m$, $U^i \lhd U^{i+1}$. Work in $V[G]$. For every $\xi < \kappa$, let $U^{\times}_{\xi}$ be the normal measure used at stage ξ in the iteration. We define an ultrafilter U^i_{ξ}: if $U^{\times}_{\xi}$ concentrates on $d''(\Delta_i \cap \xi)$, set $\underset{\sim}{U}^i_{\xi}$ to be the ultrafilter which concentrates on $\Delta_i \cap \xi$ and is projected via d onto $U^{\times}_{\xi}$. Else, set $U^i_{\xi} = U^*_{\xi}$.

Let $\underset{\sim}{\mathcal{U}}^i \in V$ be the sequence of names for the measures U^i_{ξ} defined above. Consider in $M_{U^{i+1}}$ the P_{κ}-name $j_{U^{i+1}}\left(\underset{\sim}{\mathcal{U}}^i\right)(\kappa)$, and let–

$$F = \left(j_{U^{i+1}}\left(\underset{\sim}{\mathcal{U}}^i\right)(\kappa)\right)_G \cap M_{U^{i+1}} \in M_{U^{i+1}}$$

F is a normal measure on κ which belongs to $M_{U^{i+1}}$. Thus, it suffices to prove that $F = U^i$.

Pick $X \in F$. Let $p \in G$ be a condition such that $p \Vdash X \in j_{U^{i+1}}\left(\underset{\sim}{\mathcal{U}}^i\right)(\kappa)$, namely–

$$\{\xi \in \Delta \colon p \restriction_{\xi} \Vdash X \cap \xi \in U^i_{\xi}\} \in U^{i+1}$$

we would like to argue that U^i_{ξ} in the above equation is the measure which concentrates on Δ_i and is projected via d onto $U^{\times}_{\xi}$. This requires to have–

$$\{\xi \in \Delta \colon p \restriction_{\xi} \Vdash d''(\Delta_i \cap \xi) \in U^{\times}_{\xi}\} \in U^{i+1}$$

Let us argue that p can be extended inside G such that this holds. Work over $M_{U^{i+1}}$, and extend p in G such that–

$$p \parallel \Delta_i \in j_{U^{i+1}}\left(\underset{\sim}{U}^{\times}\right)(\kappa)$$

It's enough to argue that p decides the above statement in a positive way. Assume otherwise. Then–

$$\{\xi < \kappa \colon p \restriction_{\xi} \Vdash \Delta_i \cap \xi \notin U^{\times}_{\xi}\} \in U^{i+1} \subseteq W^{i+1}$$

For every ξ in the above set (but finitely many), $d(\xi) \notin d''\Delta_i$. In particular, W^{i+1} concentrates on such ξ-s, and thus in $M[H]$, $\kappa \notin d''\Delta_i$, which is a contradiction.

Thus we can assume that $p \in G$ and–

$$\{\xi \in \Delta \colon p \restriction_{\xi} \Vdash d''(X \cap \Delta_i \cap \xi) \in U^{\times}_{\xi}\} \in U^{i+1}$$

Therefore,

$$\{\xi \in \Delta\colon p \Vdash d(\xi) \in d''(X \cap \Delta_i \cap \xi)\} \in U^{i+1}$$

and thus, in $V[G]$,

$$\{\xi \in \Delta\colon d(\xi) \in d''(X \cap \Delta_i)\} \in W^{i+1}$$

so–

$$d''\{\xi \in \Delta\colon d(\xi) \in d''(X \cap \Delta_i)\} \in W$$

so $d''(X \cap \Delta_i) \in W$, and in particular, $X \in W^i$. So $X \in U^i = W^i \cap V$.

Finally, let us argue that $U^0 \lhd U^1$. Consider in M_{U^1} the name $j_{U^1}(\underset{\sim}{\mathcal{U}}^{\times})(\kappa)$, and let $F \in M_{U^1}$ be its value with respect to the generic G. It suffices to prove that $F = U^0$. given $X \in F$, there exists $p \in G$ such that–

$$\{\xi \in \Delta\colon p\restriction_{\xi} \Vdash X \cap \xi \in U^{\times}_{\xi}\} \in U^1$$

In $V[G]$,

$$\{\xi \in \Delta\colon d(\xi) \in X\} \in W^1$$

Recall that $W^1 \equiv_{RK} W$, and thus in $M[H]$,

$$d([Id]_{W^1}) \in j_{W^1}(X) = j_W(X) = k(j_U(X))$$

where $k\colon M_U \to M$ is the embedding which satisfies $k([f]_U) = [f]_{W^*}$. Recall that $\text{crit}(k) = \kappa$, and thus $\kappa \in k(\kappa \cap j_U(X)) = k(X)$. In particular, $X \in U^0$. $\square$

Remark 3.4.5. *Denote $\langle \mu_0^{*1}, \ldots, \mu_0^{*m}\rangle = d^{-1}\{\kappa\} = \langle [Id]_{W^1}, \ldots, [Id]_{W^*}\rangle$. Then for every $1 \le i \le m-1$, $U^i = \{X \subseteq \kappa\colon \mu_0^{*i} \in k(X)\}$. Indeed, assume that $X \subseteq \kappa$ and $\mu_0^{*i} \in k(X) = k(j_U(X) \cap \kappa) = j_{W^*}(X) \cap [Id]_{W^*}$. Since W^i and W^* are Rudin-Keisler equivalent and $\mu_0^{*i} = [Id]_{W^i}$, it follows that $X \in W^i$. Therefore $X \in U^i$.*

In M_U, we can derive a measure on $[\kappa]^m$ using k as follows:

$$\mathcal{E}_0 = \{X \subseteq [\kappa]^m : \langle \kappa, \mu_0^{*1}, \ldots, \mu_0^{*m-1}\rangle \in k(X)\}$$

Corollary 3.4.6. *$\mathcal{E}_0 \in M_U$ is the product measure $U^0 \times \ldots \times U^{m-1}$ on $[\kappa]^m$, namely, for every $X \subseteq [\kappa]^m$, $X \in \mathcal{E}_0$ of and only if–*

$$\{\nu_0 < \kappa\colon \{\nu_1 < \kappa\colon \ldots \{\nu_{m-1} < \kappa\colon \langle \nu_0, \ldots, \nu_{m-1}\rangle \in X\} \in U^{m-1} \ldots\} \in U^1\} \in U^0$$

Proof. It suffices to prove that for each $X \subseteq [\kappa]^m$, $X \in U^0 \times \ldots \times U^{m-1}$ implies that $X \in \mathcal{E}_0$. Indeed, given X in the product measure, there are sets $X_0 \in U^0, \ldots, X_{m-1} \in U^{m-1}$ such that–

$$(X_0 \times \ldots \times X_{m-1}) \cap [\kappa]^m \subseteq X$$

By Remark 3.4.5, it follows that–

$$\langle \mu_0^{*1}, \ldots, \mu_0^{*m} \rangle \in k(X_0) \times \ldots \times k(X_{m-1}) \subseteq k(X)$$

as desired. □

Ult $(M_U, \mathcal{E}_0)$ is isomorphic to the finite iterated ultrapower of V, with decreasing order, with $U^0 \lhd U^1 \lhd \ldots \lhd U^{m-1} \lhd U^m$.

3.4.2 Description of the Iteration

Assume that $W \in V[G]$ is a normal measure on κ. Let $U \in V$ be a normal measure such that $W = U^{\times}$. Denote $\kappa^* = j_W(\kappa)$ (we will later prove that $\kappa^* = j_U(\kappa)$). Let $j_W : V[G] \to M[H]$ be the ultrapower embedding. We work by induction on $\alpha \leq \kappa^*$ and define an iterated ultrapower $\langle M_\alpha : \alpha \leq \kappa^* \rangle$. We define as well, for every $\alpha < \kappa^*$,

1. Elementary embeddings $j_\alpha : V \to M_\alpha$ and $k_\alpha : M_\alpha \to M$, such that $j_W \restriction_V = k_\alpha \circ j_\alpha$.

2. The ordinal $\mu_\alpha = \text{crit}(k_\alpha)$, which will turn out to be measurable in M_α.

3. A natural number $1 \leq m_\alpha < \omega$, and a sequence of normal measures on μ_α,

$$U^0_{\mu_\alpha} \lhd \ldots \lhd U^{m_\alpha - 1}_{\mu_\alpha}$$

   each of them belong to M_α. We also denote by $\mathcal{E}_\alpha$ be the measure on $[\mu_\alpha]^{m_\alpha}$ defined by taking product of the above measures, namely, a set $X \subseteq [\mu_\alpha]^{m_\alpha}$ belongs to $\mathcal{E}_\alpha$ if and only if–

$$\{\nu_{m_{\alpha-1}} < \mu_\alpha : \{\ldots \{\nu_0 < \mu_\alpha : \langle \nu_0, \ldots, \nu_{m_\alpha - 1} \in X \rangle\} \in U^{m_\alpha - 1}_{\mu_\alpha} \ldots\}\} \in U^0_{\mu_\alpha}$$

   Possibly $m_\alpha = 1$ and then $\mathcal{E}_\alpha = U^0_{\mu_\alpha}$.

Let us demonstrate the first two steps in $j_W \restriction_V$. Recall the system $W \cap V = U^0 \lhd U^1 \lhd \ldots \lhd U^m = U$. First, let $M_0 = \text{Ult}(V, U) = \text{Ult}(V, U^m)$. Let $k_0 : M_0 \to M$ be the embedding which satisfies, for every $f \in V$,

$$k_0([f]_U) = [f]_{W^*}$$

k_0 is elementary since $U \subseteq W^*$; furthermore, $\mu_0 = \text{crit}(k_0) = \kappa$. Assuming that $m = \left| j_W(d)^{-1}\{\kappa\} \right| \geq 1$, it turns out that $m_0 = m$ and $U^j_{\mu_0} = U^j$ for every $j \leq m-1$. Thus, $\mathcal{E}_0$ is the product measure $U^0 \times \ldots \times U^{m-1}$ (as defined in the previous subsection). We will then define $M_1 = \text{Ult}(M_0, \mathcal{E}_0)$. If $m = 0$, $\mu_0 = \text{crit}(k_0)$ is the first measurable above κ in M_0, $m_0 = 1$ and $\mathcal{E}_0 = U^0_{\mu_0}$.

The iteration $\langle M_\alpha \colon \alpha \leq \kappa^* \rangle$ is continuous, namely, for every limit $\alpha \leq \kappa^*$, M_α is the direct limit of $\langle M_\beta \colon \beta < \alpha \rangle$. At successor steps, $M_{\alpha+1} = \text{Ult}(M_\alpha, \mathcal{E}_\alpha)$.

For simplicity, we denote the sequence $[Id]_{\mathcal{E}_\alpha}$ by $[Id]_\alpha$. Arguing by induction, every element in M_α has the form–

$$j_\alpha(f)\left(j_{1,\alpha}(\kappa), j_{\alpha_0+1,\alpha}\left([Id]_{\alpha_0}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right) \tag{3.1}$$

for some $f \in V$ and $\alpha_0 < \ldots < \alpha_k < \alpha$.

Remark 3.4.7. *$M_{\alpha+1} = Ult(M_\alpha, \mathcal{E}_\alpha)$ can be viewed as iteration of length m_α of M_α, in the following sense: denote–*

$$M^{m_\alpha-1}_\alpha = Ult\left(M_\alpha, U^{m_\alpha-1}_{\mu_\alpha}\right)$$

$$M^{m_\alpha-2}_\alpha = Ult\left(M^{m_\alpha}_{\mu_\alpha}, U^{m_\alpha-2}_{\mu_\alpha}\right)$$

etc., up to–

$$M^0_\alpha = Ult\left(M^1_\alpha, U^0_{\mu_\alpha}\right)$$

and take $M_{\alpha+1} = M^0_\alpha$. Denote–

$$\mu^1_\alpha = j_{U^0_{\mu_\alpha}}(\mu_\alpha), \ \mu^2_\alpha = j_{U^1_{\mu_\alpha}}(\mu_\alpha), \ \ldots, \mu^{m_\alpha-1}_\alpha = j_{U^{m_\alpha-2}_{\mu_\alpha}}(\mu_\alpha)$$

Then each element in $M_{\alpha+1}$ has the form–

$$j_{\alpha,\alpha+1}(f)\left(\mu_\alpha, \mu^1_\alpha, \ldots, \mu^{m_\alpha-1}_\alpha\right)$$

for some $f \in V$, and can be identified with $[f]_{\mathcal{E}_\alpha}$. In particular, if $Id \colon [\kappa]^{m_\alpha} \to V$ is the identity function, then–

$$[Id]_{\mathcal{E}_\alpha} = \langle \mu_\alpha, \mu^1_\alpha, \ldots, \mu^{m_\alpha-1}_\alpha \rangle$$

Before we proceed, we would like to present several examples in the case where the Mitchell order is linear in V.

Example 1: Assume that the Mitchell order on each measurable is linear in V. For every $\alpha \in \Delta$, let $U_{\alpha,0}$ be the unique measure on α of order 0. Let $P = P_\kappa$ be the Magidor iteration, where, for each $\alpha \in \Delta$, the measure $U^*_{\alpha,0} = U^\times_{\alpha,0}$ is taken to be W_α. In $V[G]$, consider $W = U^*_{\kappa,0} = U^\times_{\kappa,0}$. In this case, $d''\Delta \notin W$, $m(W) = 0$ and $j_W \restriction_V$ is an iterated ultrapower of V, starting with $U_{\kappa,0}$. After this step κ is no longer measurable. Let $\alpha < \kappa^* = j_{U_{\kappa,0}}(\kappa)$. In M_α, μ_α is the least measurable $\geq \sup\{\mu_\beta \colon \beta < \alpha\}$ with cofinality above κ in V, and $M_{\alpha+1} = \text{Ult}\left(M_\alpha, U^{M_\alpha}_{\mu_\alpha,0}\right)$ is the ultrapower with the unique measure of order 0 on μ_α in M_α.

Example 2: Assume the same settings as in the previous example, but now $W = U^\times$ for arbitrary U of order higher than 0 (below κ we still assume that measures of order 0 are used). We argue that now, $m = m(W) = 1$. First, since $o(U) > 0$, $d''\Delta \in W$, and thus $d^{-1}\{\kappa\} \neq \emptyset$ in $M[H]$. So $m \geq 1$. In order to prove that $m = 1$, it suffices to prove that the following property holds in $V[G]$: There exists a finite subset $b \subseteq \kappa$ such that d is an injection on $\Delta \setminus b$. Furthermore, other then finitely many, all the Prikry sequences G adds to measurables in Δ are pairwise disjoint. Let us provide the proof. For every $\alpha \in \Delta$, let $C_\alpha \subseteq \alpha$ be the Prikry sequence added to α in $V[G]$. Then, for every $\alpha \in \Delta$,

$$\bigcup_{\beta<\alpha} C_\beta \notin U^*_{\alpha,0} = U^\times_{\alpha,0} \tag{3.2}$$

since otherwise there exists $p \in G_\alpha$ such that $\left(j_{U_{\alpha,0}}(p)\right)^{-\alpha} \Vdash \alpha \in \bigcup_{\beta<j_{U_{\alpha,0}}(\alpha)} \underset{\sim}{C}_\beta$; but $\left(j_{U_{\alpha,0}}(p)\right)^{-\alpha}$ forces that α cannot belong to Prikry sequences of measurables above α, a contradiction.

Now we can apply equation 3.2 in a density argument: Every condition p can be direct extended to $p^* \geq^* p$ by removing from each set $\underset{\sim}{A}^p_\alpha \in W_\alpha$ (where $\alpha \in \Delta$) the set $\bigcup_{\beta<\alpha} C_\beta$. Then p^* forces that the Prikry sequences added to measurables of Δ, aside from finitely many, are pairwise disjoint.

Thus $m = 1$, and the system $U^0 \lhd U^1$ consists of $U_{\kappa,0} = U^0 \lhd U^1 = U$. The first step in $j_W \restriction_V$ is $\text{Ult}(V, U)$, and $U_{\kappa,0}$ is applied ω-many times to produce a Prikry sequence of critical points to $[Id]_{W^1}$, which is the only element in $d^{-1}\{\kappa\}$. For every $\alpha < \kappa^*$, $M_{\alpha+1} = \text{Ult}\left(M_\alpha, U^{M_\alpha}_{\mu_\alpha,0}\right)$ as in the previous example. The main difference is that the length of the iteration, $\kappa^* = j_W(\kappa) = j_U(\kappa)$ is strictly higher than $j_{U_{\kappa,0}}(\kappa)$.

Example 3: Assume again linearity of the Mitchell order, but now fix in advance $m < \omega$, and assume as well that $o(\kappa) = m+1$, namely, the normal measures on κ are $U_{\kappa,0} \lhd U_{\kappa,1} \lhd \ldots \lhd U_{\kappa,m}$. We define the iteration $P = P_\kappa$ such that for every $\alpha \in \Delta$, the measure W_α is chosen as follows: If $o(\alpha) = l+1$ for some $l \leq m$, use the measure $W_\alpha = U^\times_{\alpha,l}$. In $V[G]$, let $W = U^\times_{\kappa,m}$. We argue that $m(W) = m$. We work by induction: If $m = 0$, then $d''\Delta \notin W$ and thus $m(W) = 0$. Assume that $m \geq 1$. $W^* = U^*_{\kappa,m}$ concentrates on measurables $\alpha \in \Delta$ such that $W_\alpha = U^\times_{\alpha,m-1}$, and for each such α, $m(W_\alpha) = m-1$. So $W^* = U^*_{\kappa,m}$ concentrates on measurables $\alpha \in \Delta$ such that α is the m-th element in $d^{-1}\{d(\alpha)\}$, and thus $m(W) = m$. By linearity of the Mitchell order, the system $U^0 \lhd \ldots \lhd U^m$ is exactly the sequence $U_{\kappa,0} \lhd U_{\kappa,1} \lhd \ldots \lhd U_{\kappa,m}$. $d^{-1}\{\kappa\} = \{[Id]_{W^1}, \ldots, [Id]_{W^m}\}$ contains exactly m elements, and each $[Id]_{W^i}$ (where $1 \le i \le m$) has Prikry sequence in $M[H]$ which is generated by iterating the measure $U_{\kappa,i-1}$ ω-many times.

We would like to define the embedding $k_\alpha \colon M_\alpha \to M$. We do this assuming that embeddings $k_\beta \colon M_\beta \to M$ have been defined for every $\beta < \alpha$. We also assume by induction that for each such $\beta < \alpha$, a sequence $\vec{\mu}^*_\beta = \langle \mu^{*0}_\beta, \ldots, \mu^{*m_\beta - 1}_\beta \rangle$ has been defined. We then define $k_\alpha \colon M_\alpha \to M$ as follows:

$$k_\alpha \left(j_\alpha(f) \left(j_{0,\alpha}(\kappa), j_{\alpha_0+1,\alpha}\left([Id]_{\alpha_0}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right)\right) = j_W(f)\left([Id]_{W^*}, \vec{\mu}^*_{\alpha_0}, \ldots, \vec{\mu}^*_{\alpha_k}\right)$$

for every $f \in V$ and $1 \le \alpha_0 < \ldots < \alpha_k$.

We will prove by induction on $\alpha \le \kappa^*$ that the following properties hold:

(A) $k_\alpha \colon M_\alpha \to M$ is elementary.

(B) Denote $\mu_\alpha = \text{crit}(k_\alpha)$. Then μ_α is measurable in M_α. Moreover, μ_α is the least measurable $\mu \in M_\alpha$ which is greater or equal to $\sup\{\mu_\beta \colon \beta < \alpha\}$ and satisfies $(\text{cf}(\mu))^V > \kappa$.

(C) Let $\mu^*_\alpha = k_\alpha(\mu_\alpha)$. Then μ_α appears as an element in the Prikry sequence of $k_\alpha(\mu_\alpha)$ in H. We will denote by t_α the initial segment of the Prikry sequence of $k_\alpha(\mu_\alpha)$ below μ_α, and by n_α the length of t_α.

(D) Let $\{\mu^{*1}_\alpha, \ldots, \mu^{*m_\alpha - 1}_\alpha\}$ be the increasing enumeration of $d^{-1}(\mu_\alpha)$ below μ^*_α, and denote as well $\mu^{*0} = \mu_\alpha, \mu^*_\alpha = \mu^{m_\alpha}_\alpha$ (possibly $m_\alpha = 1$ and then μ_α does not appear as first element in Prikry sequences of measurables below μ^*_α). For every $0 \le j \le m_\alpha$, there exists a measure $U^j_{\mu_\alpha} \in M_\alpha$ on μ_α, which satisfies–

$$k_\alpha\left(U^j_{\mu_\alpha}\right) = j_W\left(\xi \mapsto U^j_\xi\right)\left(k_\alpha(\mu_\alpha)\right)$$

Moreover,

$$U^0_{\mu_\alpha} \lhd U^1_{\mu_\alpha} \lhd \ldots \lhd U^{m_\alpha - 1}_{\mu_\alpha}$$

(E) The measure $\mathcal{E}_\alpha$ which corresponds to $U^0_{\mu_\alpha} \lhd U^1_{\mu_\alpha} \lhd \ldots \lhd U^{m_\alpha - 1}_{\mu_\alpha}$ is derived from $k_\alpha \colon M_\alpha \to M$ in the following sense:

$$\mathcal{E}_\alpha = \{X \subseteq [\mu_\alpha]^{m_\alpha} : \langle \mu^{*0}_\alpha, \mu^{*1}_\alpha, \ldots, \mu^{*m_\alpha - 1}_\alpha \rangle \in k_\alpha(X)\} \cap M_\alpha$$

The proof of the above properties goes by induction on α. For $\alpha = 0$, $k_0 \colon M_0 = M_U \to M$ is the embedding which maps each $[f]_U$ to $[f]_{W^*}$; it has critical point $\mu_0 = \kappa$. In $M[H]$, μ_0 appears as a first element in the Prikry sequence of $k_0(\mu_0) = [Id]_{W^*}$, and of $m-1$ measurables ${\mu_0}^{*1} = [Id]_{W^1}, \ldots, {\mu_0}^{*m-1} = [Id]_{W^{m-1}}$. The measure $\mathcal{E}_0 \in M_0$ derived from k_0 using $\langle \mu_0, {\mu_0}^{*1}, \ldots, {\mu_0}^{*m-1} \rangle$ is indeed the product of $U^0 \lhd \ldots \lhd U^{m-1}$ by remark 3.4.5.

We proceed and prove the properties for arbitrary $0 < \alpha < \kappa^*$.

Lemma 3.4.8. *$k_\alpha \colon M_\alpha \to M$ is elementary, and $j_W \restriction_V = k_\alpha \circ j_\alpha$.*

Proof. For $\alpha = 0$, we already argued that $k_0 \colon M_0 \to M$ in elementary.

For simplicity, we will prove that for every $x, y \in M_\alpha$, $M_\alpha \vDash x \in y$ if and only if $M \vDash k_\alpha(x) \in k_\alpha(y)$.

Let us focus on the case where $\alpha = \alpha' + 1$ is successor, as the limit case is simpler. There are functions $f, g \in V$ and $\alpha_0 < \ldots < \alpha_k < \alpha'$ such that–

$$x = j_\alpha(f)\left(j_{0,\alpha}(\kappa), j_{\alpha_0+1,\alpha}\left([Id]_{\alpha_0}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right), j_{\alpha'+1,\alpha}\left([Id]_{\alpha'}\right)\right)$$

$$y = j_\alpha(g)\left(j_{0,\alpha}(\kappa), j_{\alpha_0+1,\alpha}\left([Id]_{\alpha_0}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right), j_{\alpha'+1,\alpha}\left([Id]_{\alpha'}\right)\right)$$

We assumed that $M_\alpha = \mathrm{Ult}\left(M_{\alpha'}, \mathcal{E}_{\alpha'}\right) \vDash x \in y$, namely,

$$\mathrm{Ult}\left(M_{\alpha'}, \mathcal{E}_{\alpha'}\right) \vDash [Id]_{\alpha'} \in j_{\alpha',\alpha}(X)$$

where X is the set–

$$\Big\{\vec{\xi} \colon\ j_{\alpha'}(f)\Big(j_{0,\alpha'}(\kappa), j_{\alpha_0+1,\alpha'}\left([Id]_{\alpha_0}\right), \ldots, j_{\alpha_k+1,\alpha'}\left([Id]_{\alpha_k}\right), \vec{\xi}\Big) \in j_{\alpha'}(g)\Big(j_{0,\alpha'}(\kappa), j_{\alpha_0+1,\alpha'}\left([Id]_{\alpha_0}\right), \ldots, j_{\alpha_k+1,\alpha'}\left([Id]_{\alpha_k}\right), \vec{\xi}\Big)\Big\}$$

In particular, $X \in \mathcal{E}_{\alpha'}$, and thus $\vec{\mu}^*_{\alpha'} \in k_{\alpha'}(X)$. Since $j_W \restriction_V = k_{\alpha'} \circ j_{\alpha'}$, it follows that–

$$j_W(f)\left([Id]_{W^*}, \vec{\mu}^*_{\alpha_0}, \ldots, \vec{\mu}^*_{\alpha_k}, \vec{\mu}^*_{\alpha'}\right) \in j_W(g)\left([Id]_{W^*}, \vec{\mu}^*_{\alpha_0}, \ldots, \vec{\mu}^*_{\alpha_k}, \vec{\mu}^*_{\alpha'}\right)$$

namely $k_\alpha(x) \in k_\alpha(y)$. □

We will present the proof of properties (B)-(E) is the next subsections.

3.4.3 Multivariable Fusion

Assume from now on that $\alpha > 0$ is fixed, and we are at stage α in the inductive proof of properties (A)-(E). In this subsection we develop a generalization of lemma 3.2.3 (the Fusion lemma).

Since $\alpha > 0$, we may assume in equation 3.1 that $\alpha_0 = 0$. This will simplify some of the arguments below. We can also denote $\mathcal{E}_0' = U^0 \times \ldots \times U^{m-1} \times U^m$ (including U^m, unlike $\mathcal{E}_0$) and $[Id]_0' = [Id]_0{}^\frown j_{0,1}(\kappa)$ so that every element in M_α has the form–

$$j_\alpha(f)\left(j_{1,\alpha}\left([Id]_0'\right), j_{\alpha_1+1,\alpha}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right) \tag{3.3}$$

for some $f \in V$ and $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k < \alpha$.

Definition 3.4.9. *Let $p \in P_\kappa$ be a condition and $m \geq 1$. We define, by induction, when an increasing sequence $\langle \xi, \xi^1, \ldots, \xi^m \rangle$ below κ is admissible for p. In case it is, we also define an extension $p^\frown\langle \xi, \xi^1, \ldots, \xi^m \rangle \geq p$.*
Intuitively, $\langle \xi, \xi^1, \ldots, \xi^m \rangle$ is admissible for p if p can be extended (in a specific way, described below) to a condition $p^\frown\langle \xi, \xi^1, \ldots, \xi^m \rangle$ which forces that $d^{-1}\{\xi\} = \{\xi^1, \ldots, \xi^m\}$.
We provide the definition under the assumption $p \geq^* 0$. Else, consider only sequences $\langle \xi, \xi^1, \ldots, \xi^m \rangle$ such that ξ is an upper bound of the finite set of the coordinates $\beta < \kappa$ in which $p(\beta)$ non-directly extends $0_{\underset{\sim}{Q}_\beta}$.

1. *$\langle \xi, \xi^1 \rangle$ is admissible for p if $\left(p\restriction_{\xi^1}\right)^{-\xi} \Vdash \xi \in \underset{\sim}{A}^p_{\xi^1}$. In this case, we define–*

$$p^\frown\langle \xi, \xi^1 \rangle = \left(\left(p\restriction_{\xi^1}\right)^{-\xi}\right)^\frown \langle \langle \check{\xi} \rangle, \underset{\sim}{A}^p_{\xi^1} \rangle^\frown p \setminus \left(\xi^1 + 1\right)$$

2. *Assume that $1 \leq i \leq m-1$ and $\langle \xi, \xi^1, \ldots, \xi^i \rangle$ is admissible for p. Assume also that $q = p^\frown\langle \xi, \xi^1, \ldots, \xi^i \rangle$ has been defined. Then $\langle \xi, \xi^1, \ldots, \xi^i, \xi^{i+1} \rangle$ is admissible for p if $q^{-\xi_i}\restriction_{\xi^{i+1}} \Vdash \xi \in \underset{\sim}{A}^p_{\xi^{i+1}}$. In this case, we define–*

$$p^\frown\langle \xi, \xi^1, \ldots, \xi^i, \xi^{i+1} \rangle = \left(q^{-\xi_i}\restriction_{\xi^{i+1}}\right)^\frown \langle \langle \check{\xi} \rangle, \underset{\sim}{A}^p_{\xi^{i+1}} \rangle^\frown p \setminus \left(\xi^{i+1} + 1\right)$$

   In the case where $i = m-1$, we make a minor change in the above definition and set–

$$p^\frown\langle \xi, \xi^1, \ldots, \xi^m \rangle = \left(q^{-\xi_{m-1}}\restriction_{\xi^m}\right)^\frown \langle \langle \check{\xi} \rangle, \underset{\sim}{A}^p_{\xi^m} \rangle^\frown \left(p \setminus \left(\xi^m + 1\right)\right)^{-\xi^m}$$

   (namely, remove $\xi^m + 1$ from large sets in places above ξ^m).

In other words, if $\vec{\xi} = \langle \xi, \xi^1, \ldots, \xi^m \rangle$ is admissible for p, then we set–

$$p^\frown \langle \vec{\xi} \rangle = \left(p \restriction_{\xi^1}\right)^{-\xi^\frown} \langle \langle \check{\xi} \rangle, \underset{\sim}{A}^p_{\xi^1} \rangle^\frown \left(p \restriction_{(\xi^1, \xi^2)}\right)^{-\xi^{1\frown}} \langle \langle \check{\xi} \rangle, \underset{\sim}{A}^p_{\xi^2} \rangle^\frown \ldots^\frown$$
$$\left(p \restriction_{(\xi^{m-1}, \xi^m)}\right)^{-\xi^{m-1\frown}} \langle \langle \check{\xi} \rangle, \underset{\sim}{A}^p_{\xi^m} \rangle^\frown \left(p \setminus (\xi^m + 1)\right)^{-\xi^m}$$

In $V[G]$, denote, for every $\xi \in d''\Delta$ with $\left|d^{-1}\{\xi\}\right| = m$, $d^{-1}\{\xi\} = \langle \mu_0^{*1}(\xi), \ldots, \mu_0^{*m}(\xi) \rangle = \vec{\mu}_0^*(\xi)$. Then in $M[H]$, the sequence $[\xi \mapsto \vec{\mu}_0^*(\xi)]_W$ is–

$$\langle \kappa, \mu_0^{*1}, \ldots, \mu_0^{*m-1}, \mu_0^{*m} \rangle = \langle [Id]_W, [Id]_{W^1}, \ldots, [Id]_{W^{m-1}}, [Id]_{W^m} \rangle$$

Theorem 3.4.10. *Let $p \in P_\kappa$. For every increasing $\vec{\xi} = \langle \xi, \xi^1, \ldots, \xi^m \rangle$, let $e\left(\vec{\xi}\right)$ be a P_ξ-name for a subset of $P \setminus \xi$ which is $\leq^*$-dense open above conditions which force that $d^{-1}\{\xi\} = \langle \xi^1, \ldots, \xi^m \rangle$. Then there exists $p^* \geq^* p$ and a set $X \in U^0 \times U^1 \times \ldots \times U^m$ such that for every increasing $\langle \xi, \xi^1, \ldots, \xi^m \rangle \in X$ which is admissible for p^*,*

$$p^* \restriction_\xi \Vdash p^{*\frown} \langle \xi, \xi^1, \ldots, \xi^m \rangle \setminus \xi \in e\left(\xi, \xi^1, \ldots, \xi^m\right)$$

Furthermore, if p^* as above is chosen in G, then $U^0 \times U^1 \times \ldots \times U^m$ concentrates on the set of admissible sequences for p^*, and–

$$\{\xi < \kappa \colon \langle \xi, \mu_0^{*1}(\xi), \ldots, \mu_0^{*m}(\xi) \rangle \text{ is admissible for } p^*,\ p^{*\frown} \langle \xi, \mu_0^{*1}(\xi), \ldots, \mu_0^{*m}(\xi) \rangle \in G$$
$$\text{and } p^* \restriction_\xi \Vdash p^{*\frown} \langle \xi, \mu_0^{*1}(\xi), \ldots, \mu_0^{*m}(\xi) \rangle \setminus \xi \in e\left(\xi, \mu_0^{*1}(\xi), \ldots, \mu_0^{*m}(\xi)\right)\} \in W$$

Proof. Assume for simplicity that $p \geq^* 0$. Else, just work with values of ξ above some ordinal μ for which $p \setminus \mu \geq^* 0$.

Let us first sketch the main steps of the proof. We will first define, for every sequence $\vec{\xi} = \langle \xi, \xi^1, \ldots, \xi^m \rangle$, a condition $p\left(\vec{\xi}\right) = p\left(\xi, \xi^1, \ldots, \xi^m\right) \geq^* p$. We define it such that for every $1 \leq i < m$, if $\langle \xi, \xi^1, \ldots, \xi^i \rangle$ is admissible for $p\left(\vec{\xi}\right)$,

$$p\left(\vec{\xi}\right)^\frown \langle \xi, \xi^1, \ldots, \xi^i \rangle \restriction_{\xi^{i+1}} \Vdash \xi \in \underset{\sim}{A}^{p\left(\vec{\xi}\right)}_{\xi^{i+1}}$$

This can be done in a trivial way, by taking a direct extension which removes ξ from the measure one sets at the relevant coordinate; we will avoid such trivialities by shrinking the measure one sets only above $\xi + 1$ (namely, instead of shrinking a large set A to a set B, shrink it to $(A \cap (\xi + 1)) \cup (B \setminus (\xi + 1))$).

Once $p\left(\vec{\xi}\right)$ is defined, we define a condition $r\left(\vec{\xi}\right) \in P \setminus \xi$: If the sequence $\vec{\xi}$ is admissible for $p\left(\vec{\xi}\right)$, we take $r\left(\vec{\xi}\right) \geq^* \left(p\left(\vec{\xi}\right)^\frown \langle\vec{\xi}\rangle\right) \setminus \xi$, with $r\left(\vec{\xi}\right) \in e\left(\vec{\xi}\right)$. Else, take $r\left(\vec{\xi}\right) = p\left(\vec{\xi}\right) \setminus \xi$.

The second step will be to define, for every initial segment $\langle \xi, \xi^1, \ldots, \xi^i \rangle$ of $\langle \xi, \xi^1, \ldots, \xi^m \rangle$, a condition $r\left(\xi, \xi^1, \ldots, \xi^i\right) \in P \setminus \xi$, such that the family $\langle r\left(\xi, \xi^1, \ldots, \xi^i\right) : i \leq m \rangle$ is coherent in the following sense: There exists a set $X \in U^0 \times U^1 \times \ldots \times U^m$ such that, for every $\vec{\xi} \in X$ and for every $1 \leq i < j \leq m$,

$$r\left(\xi, \xi^1, \ldots, \xi^i\right) \restriction_{\xi^{i+1}} = r\left(\xi, \xi^1, \ldots, \xi^j\right) \restriction_{\xi^{i+1}}$$

The set X obtained in this step will be the set X from the formulation of the lemma. Since X belongs to the product measure, we can fix sets $X_0 \in U^0, \ldots, X_m \in U^m$ such that–

$$(X_0 \times X_1 \times \ldots \times X_m) \cap [\kappa]^m \subseteq X$$

The third step will be to plug together all the conditions $r\left(\xi, \xi^1, \ldots, \xi^i\right)$, $i \leq m$. We will do this step by step, by constructing a sequence of direct extensions of the original condition p,

$$p \leq^* p^0 \leq^* p^1 \leq^* \ldots \leq^* p^m$$

where each p^i has the following property: For every increasing sequence $\langle \xi, \xi^1, \ldots, \xi^i \rangle \in X_0 \times \ldots \times X_i$ which is admissible for p^i,

$$\left((p^i)^\frown \langle \xi, \xi^1, \ldots, \xi^i \rangle\right)^{-\xi^i} \geq r\left(\xi, \xi^1, \ldots, \xi^i\right)$$

Eventually, the condition $p^* = p^m$ will be as required in the formulation of the theorem.

The fourth and final step will be the proof of the ”furthermore” part in the formulation of the theorem.

Step 1: Construction of $p\left(\vec{\xi}\right) \in P$ and $r\left(\vec{\xi}\right) \in P \setminus \xi$. Fix a sequence $\vec{\xi} = \langle \xi, \xi^1, \ldots, \xi^m \rangle$. We construct $p\left(\vec{\xi}\right) \geq^* p$. Work in the forcing $P \restriction_{[\xi^{m-1}, \xi^m)}$, above a generic extension for $P \restriction_{\xi^{m-1}}$ which contains $p \restriction_{\xi^{m-1}}$. We choose $p\left(\vec{\xi}\right) \restriction_{[\xi^{m-1}, \xi^m)} \geq^* p \restriction_{[\xi^{m-1}, \xi^m)}$ such that:

$$A^{p\left(\vec{\xi}\right)}_{\xi^{m-1}} = \left(A^p_{\xi^{m-1}} \cap (\xi+1)\right) \cup (B \setminus (\xi+1))$$

and–

$$p\left(\vec{\xi}\right) \restriction_{(\xi^{m-1}, \xi^m)} = r$$

where $B \in W_{\xi^{m-1}}$ and $r \in P \restriction_{(\xi^{m-1}, \xi^m)}$ are chosen such that $\langle \langle \xi \rangle, B \rangle^\frown r \parallel \xi \in \underset{\sim}{A}^p_{\xi^m}$ (in the forcing $P \restriction_{[\xi^{m-1}, \xi^m)}$).

Now work in $P\restriction_{[\xi^{m-2},\xi^{m-1})}$. We choose $p\left(\vec{\xi}\right)\restriction_{[\xi^{m-2},\xi^{m-1})}\geq^* p\restriction_{[\xi^{m-2},\xi^{m-1})}$ in a similar manner:

$$A^{p\left(\vec{\xi}\right)}_{\xi^{m-2}}=\left(A^p_{\xi^{m-2}}\cap(\xi+1)\right)\cup(B\setminus(\xi+1))$$

and–

$$p\left(\vec{\xi}\right)\restriction_{(\xi^{m-1},\xi^m)}=r$$

where $B\in W_{\xi^{m-2}}$ and $r\in P\restriction_{(\xi^{m-2},\xi^{m-1})}$ are chosen such that $\langle\langle\xi\rangle,B\rangle^\frown r\parallel\xi\in\underset{\sim}{A}^p_{\xi^{m-1}}$, and also decide in which way $p\left(\vec{\xi}\right)\restriction_{[\xi^{m-1},\xi^m)}$ decides the statement $\xi\in\underset{\sim}{A}^p_{\xi^m}$.

Continue in this fashion, direct extending p in the intervals $\left[\xi^i,\xi^{i+1}\right)$, shrinking $\underset{\sim}{A}^p_{\xi^i}$ only above $\xi+1$, and deciding how $p\restriction_{[\xi^{i+1},\xi^j)}{}^\frown\langle\xi,\xi^{i+1},\ldots,\xi^j\rangle$ decides the statement $\xi\in\underset{\sim}{A}^p_{\xi^j}$, for every $j\geq i+1$; By our construction, it actually decides the statement $\xi\in\underset{\sim}{A}^{p\left(\vec{\xi}\right)}_{\xi^j}$, since the large sets were shrinked only above $\xi+1$.

This produces the desired condition $p\left(\vec{\xi}\right)\geq^* p$. Now we define the conditions $r\left(\vec{\xi}\right)\in P\setminus\xi$ as above.

Step 2: Construction of the conditions $r\left(\xi,\xi^1,\ldots,\xi^j\right)\in P\setminus\xi$ for every $j\leq m$. Given $j<m$ and $\langle\xi,\xi^1,\ldots,\xi^j\rangle$, let–

$$r\left(\xi,\xi^1,\ldots,\xi^j\right)=\left[\langle\xi^{j+1},\ldots,\xi^m\rangle\mapsto r\left(\xi,\xi^1,\ldots,\xi^j,\xi^{j+1},\ldots,\xi^m\right)\restriction_{\xi^{j+1}}\right]_{U^{j+1}\times\ldots\times U^m}$$

Fix such j and $\langle\xi,\xi^1,\ldots,\xi^j\rangle$. Since all the measures $U^k,k\leq j$, are normal measures on κ, there are sets $X^j_{j+1}\left(\xi,\xi^1,\ldots,\xi^j\right)\in U^{j+1},\ X^j_m\left(\xi,\xi^1,\ldots,\xi^j\right)\in U^m$ such that for every increasing sequence $\langle\xi^{j+1},\ldots,\xi^m\rangle\in X^j_{j+1}\left(\xi,\xi^1,\ldots,\xi^j\right)\times\ldots\times X^j_m\left(\xi,\xi^1,\ldots,\xi^j\right)$,

$$r\left(\xi,\xi^1,\ldots,\xi^j\right)\restriction_{\xi^{j+1}}=r\left(\xi,\xi^1,\ldots,\xi^m\right)\restriction_{\xi^{j+1}}$$

Define, for every $k\leq m$,

$$X_k=\bigcap_{j<k}\left(\underset{\langle\xi,\xi^1,\ldots,\xi^j\rangle}{\triangle}X^j_k\left(\xi,\xi^1,\ldots,\xi^j\right)\right)\in U^k$$

Namely, $\xi^k\in X_k$ if and only if, for every $j<k$ and increasing sequence $\langle\xi,\xi^1,\ldots,\xi^j\rangle$ below ξ^k, $\xi^k\in X^j_k\left(\xi,\xi^1,\ldots,\xi^j\right)$.

Then $X_0\in U^0,\ldots,X_m\in U^m$ satisfy that for every $j\leq m$, and for every increasing sequence $\langle\xi,\xi^1,\ldots,\xi^m\rangle\in(X_0\times\ldots\times X_m)\cap[\kappa]^m$,

$$\langle\xi^{j+1},\ldots,\xi^m\rangle\in X^j_{j+1}\left(\xi,\xi^1,\ldots,\xi^j\right)\times\ldots\times X^j_m\left(\xi,\xi^1,\ldots,\xi^j\right)$$

and thus–

$$r\left(\xi,\xi^1,\ldots,\xi^j\right)\restriction_{\xi^{j+1}}= r\left(\xi,\xi^1,\ldots,\xi^m\right)\restriction_{\xi^{j+1}}$$

as desired.

Step 3: Construction of the sequence $p \leq^* p^0 \leq^* \ldots \leq^* p^m$.

We first construct $p^0 \geq^* p$. Recall that for every $\xi \in X_0$, a condition $r(\xi) \in P \setminus \xi$ is defined such that $r(\xi) \geq^* p \setminus \xi$. By the Fusion lemma 3.2.3, we can choose $p^0 \geq^* p$ such that for every $\xi \in X_0$, $p^0 \restriction_\xi \Vdash p^0 \setminus \xi \geq^* q(\xi)$.

Assume that $i < m$ and p^i has been constructed such that, for every $\langle \xi,\xi^1,\ldots,\xi^i\rangle \in X_0\times\ldots\times X_i$ which is admissible to it,

$$p^i \restriction_\xi \Vdash \left(\left(p^i\right)^\frown \langle \xi,\xi^1,\ldots,\xi^i\rangle \setminus \xi\right)^{-\xi^i} \geq r\left(\xi,\xi^1,\ldots,\xi^i\right)$$

We now construct $p^{i+1} \geq^* p^i$. We will define, for every $\xi^{i+1} \in X_{i+1}$, a direct extension $q\left(\xi^{i+1}\right) \geq^* p^i \setminus \xi^{i+1}$. $p^{i+1} \geq^* p^i$ will be generated from the conditions $\langle q\left(\xi^{i+1}\right) : \xi^{i+1} \in X_{i+1}\rangle$ using the Fusion lemma 3.2.3. Fix $\xi^{i+1} \in X_{i+1}$ and work in the quotient forcing $P \setminus \xi^{i+1}$. For every $\xi \in A^{p^i}_{\xi^{i+1}} \cap X_0$, we define a direct extension–

$$\langle\langle\xi\rangle, B_\xi\rangle^\frown s_\xi \geq^* \langle\langle\xi\rangle, A^{p^i}_{\xi^{i+1}} \setminus (\xi+1)\rangle^\frown p^i \setminus \left(\xi^{i+1}+1\right)$$

such that, if–

1. The increasing enumeration of $d^{-1}\{\xi\} \cap \xi^{i+1}$ is a sequence $\langle \xi^1,\ldots,\xi^i\rangle$ of length i;

2. $\langle \xi^1,\ldots,\xi^i\rangle \in X_1\times\ldots\times X_i$;

3. $r\left(\xi,\xi^1,\ldots,\xi^i,\xi^{i+1}\right)\restriction_{\xi^{i+1}}$ belongs to the generic extension up to coordinate ξ^{i+1};

then $\langle\langle\xi\rangle, B_\xi\rangle^\frown s_\xi \geq r\left(\xi,\xi^1,\ldots,\xi^i,\xi^{i+1}\right)\setminus \xi^{i+1}$. Such B_ξ, r_ξ can be chosen since $r\left(\xi,\xi^1,\ldots,\xi^{i+1}\right)\setminus \xi^{i+1}$ is an extension of $p \setminus \xi^{i+1}$ which is obtained by direct extending after appending ξ to $t^p_{\xi^{i+1}}$ (if possible). Finally, take–

$$q(\xi^{i+1}) = \langle\langle\rangle, \underset{\sim}{A}^{p^i}_{\xi^{i+1}} \cap \left(\underset{\xi<\xi^{i+1}}{\triangle} B_\xi\right)\rangle^\frown s_{d(\xi^{i+1})}$$

This concludes the construction of p^{i+1}. Let us show that for every $\langle \xi,\xi^1,\ldots,\xi^{i+1}\rangle \in X_0\times\ldots\times X_{i+1}$ which is admissible for p^{i+1},

$$p^{i+1}\restriction_\xi \Vdash \left(\left(p^{i+1}\right)^\frown \langle \xi,\xi^1,\ldots,\xi^{i+1}\rangle \setminus \xi\right)^{-\xi^{i+1}} \geq r\left(\xi,\xi^1,\ldots,\xi^{i+1}\right)$$

Pick such a sequence $\langle \xi, \xi^1, \ldots, \xi^i, \xi^{i+1} \rangle$. By induction, since $p^{i+1} \geq^* p^i$ and $\langle \xi, \xi^1, \ldots, \xi^i \rangle \in X_0 \times \ldots \times X_i$ is admissible for p^{i+1},

$$p^{i+1} \restriction_\xi \Vdash \left(\left(p^{i+1}\right)^\frown \langle \xi, \xi^1, \ldots, \xi^i \rangle \restriction_{[\xi, \xi^{i+1})} \right)^{-\xi^i} \geq r\left(\xi, \xi^1, \ldots, \xi^i\right) \restriction_{\xi^{i+1}} = r\left(\xi, \xi^1, \ldots, \xi^i, \xi^{i+1}\right) \restriction_{\xi^{i+1}}$$

Now, work in a generic extension $G_{\xi^{i+1}} \subseteq P_{\xi^{i+1}}$ which contains $\left(p^{i+1}\right)^\frown \langle \xi, \xi^1, \ldots, \xi^{i+1} \rangle \restriction_{\xi^{i+1}}$. By fusion, $p^{i+1} \restriction_{\xi^{i+1}}$ forces that $\left(p^{i+1} \setminus \xi^{i+1}\right)^{-\xi^{i+1}}$ extends $q\left(\xi^{i+1}\right)$. By the above formula, $r\left(\xi, \xi^1, \ldots, \xi^{i+1}\right) \restriction_{\xi^{i+1}} \in G_{\xi^{i+1}}$. Thus, by the choice of $q\left(\xi^{i+1}\right)$,

$$\left(\langle \langle \xi \rangle, A^{p^{i+1}} \setminus \xi + 1 \rangle^\frown p^{i+1} \setminus \left(\xi^{i+1} + 1\right) \right)^{-\xi^{i+1}} \geq r\left(\xi, \xi^1, \ldots, \xi^{i+1}\right) \setminus \xi^{i+1}$$

This is true for every generic $G_{\xi^{i+1}}$ which contains $\left(p^{i+1}\right)^\frown \langle \xi, \xi^1, \ldots, \xi^{i+1} \rangle \restriction_{\xi^{i+1}}$, so–

$$p^{i+1} \restriction_\xi \Vdash \left(\left(p^{i+1}\right)^\frown \langle \xi, \xi^1, \ldots, \xi^{i+1} \rangle \setminus \xi \right)^{-\xi^{i+1}} \geq r\left(\xi, \xi^1, \ldots, \xi^{i+1}\right)$$

as desired.

Step 4: The "furthermore" part in the formulation of the theorem. Denote $p^* = p^m$ and assume that $p^* \in G$. p^* satisfies that for every increasing sequence $\langle \xi, \xi^1, \ldots, \xi^m \rangle \in X_0 \times \ldots \times X_m$, and for every $1 \leq i \leq m$,

$$\left(p^{*\frown} \langle \vec{\xi} \rangle \right)^{-\xi^i} \restriction_{\xi^i} \| \ \xi \in \underset{\sim}{A}^{p^*}_{\xi^i} \tag{3.4}$$

Let us argue that–

$$\{\xi < \kappa \colon \langle \vec{\mu}^*_0(\xi) \rangle \text{ is admissible for } p^*\} \in W$$

Take $X \in W$ such that $X \subseteq X_0 \cap d'' X_1 \cap \ldots \cap d'' X_m$. For every $\xi \in X$, $\vec{\mu}^*_0(\xi) \in X_0 \times X_1 \times \ldots \times X_m$. Then X can be shrinked to a set in W for which the decisions in equation 3.4 are positive when substituting $\vec{\xi} = \vec{\mu}^*_0(\xi)$: Indeed, otherwise, in $M[H]$, it would not hold that $d^{-1}\{\kappa\} = \langle \mu^{*1}_0, \ldots, \mu^{*m}_0 \rangle$.

Let us verify that $\{\xi < \kappa \colon p^{*\frown} \langle \vec{\mu}^*_0(\xi) \rangle \in G\} \in W$. We need to verify that for a set of ξ-s in W the following holds: for every $1 \leq i \leq m$ and for every measurable $\mu \in \left(\mu^{*i-1}(\xi), \mu^{*i}(\xi)\right)$, $d(\mu) > \mu^{*i-1}(\xi)$.

Recall the following property from the proof of lemma 3.3.12: If $d'' \Delta \in W$ (namely $m > 0$; if $m = 0$ there is nothing to prove), then there exists a finite subset $b \subseteq \kappa$ such that for every measurable $\mu > \sup(b)$,

$$d(\mu) \notin \bigcup_{\xi \in \Delta \cap \mu} (d(\xi), \xi]$$

from now on, we consider values of ξ above $\sup(b)$, such that $d^{-1}\{\xi\}$ contains only measurables μ for which the above holds (the set of such ξ-s is clearly in W). Let $\mu \in \left(\mu^{*i-1}(\xi), \mu^{*i}(\xi)\right)$. First, note that $\mu < \mu^{*i}_{\xi}$, and thus–

$$\xi = d\left(\mu^{*i}(\xi)\right) \notin (d(\mu), \mu]$$

so $d(\mu) \geq \xi$, and thus $d(\mu) > \xi$. This also proves the desired property for $i = 1$. Assume now that $i > 1$. Then $\mu > \mu^{*i-1}(\xi)$ and thus–

$$d(\mu) \notin \left(d\left(\mu^{*i-1}(\xi)\right), \mu^{*i-1}(\xi)\right] = \left(\xi, \mu^{*i-1}(\xi)\right]$$

since we already proved that $d(\mu) > \xi$, it follows that $d(\mu) > \mu^{*i-1}(\xi)$, as desired. □

Recall that, given $\beta < \alpha$, μ_β appears as an element in the Prikry sequence of $k_\beta(\mu_\beta)$. Also, t_β is the initial segment of this sequence, consisting of all the ordinals below μ_β; we also denote $n_\beta = \mathrm{lh}(t_\beta)$. Finally, there exists a natural number $m_\beta < \omega$ and a corresponding sequence of measures,

$$U^0_{\mu_\beta} \lhd U^1_{\mu_\beta} \lhd \ldots \lhd U^{m_\beta - 1}_{\mu_\beta}$$

each of them belong to M_β.

We would like to construct, in V, functions which represent $\mu_\beta, t_\beta, U^j_{\mu_\beta}$ $(0 \leq j < m_\beta)$ in $\mathrm{Ult}(V[G], W)$. To do this, we first need to understand how the same objects are represented in the iterated ultrapower $j_\beta \colon V \to M_\beta$, in the sense of the following definition.

Definition 3.4.11. *Fix $0 < \alpha \leq \kappa^*$. An increasing sequence $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k$ below α is called nice, if the are functions g_i, f_i, F^j_i for every $1 \leq i \leq k$ and $0 \leq j \leq m_{\alpha_i}$ such that–*

$$\mu_{\alpha_1} = j_{\alpha_1}(g_1)\left(j_{1,\alpha_1}\left([Id]'_0\right)\right)$$

$$t_{\alpha_1} = j_{\alpha_1}(f_1)\left(j_{1,\alpha_1}\left([Id]'_0\right)\right)$$

$$U^j_{\mu_{\alpha_1}} = j_{\alpha_1}\left(F^j_1\right)\left(j_{1,\alpha_1}\left([Id]'_0\right)\right) \quad (0 \leq j < m_1)$$

and, for every $1 \leq i < k$,

$$\mu_{\alpha_{i+1}} = j_{\alpha_{i+1}}(g_{i+1})\left(j_{1,\alpha_1}\left([Id]'_0\right), j_{\alpha_1,\alpha_{i+1}}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_i,\alpha_{i+1}}\left([Id]_{\alpha_i}\right)\right)$$

$$t_{\alpha_{i+1}} = j_{\alpha_{i+1}}(f_{i+1})\left(j_{1,\alpha_1}\left([Id]'_0\right), j_{\alpha_1,\alpha_{i+1}}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_i,\alpha_{i+1}}\left([Id]_{\alpha_i}\right)\right)$$

$$U^j_{\mu_{\alpha_{i+1}}} = j_{\alpha_{i+1}}\left(F^j_{i+1}\right)\left(j_{1,\alpha_1}\left([Id]'_0\right), j_{\alpha_1,\alpha_{i+1}}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_i,\alpha_{i+1}}\left([Id]_{\alpha_i}\right)\right) \quad \left(0 \leq j < m_{\alpha_{i+1}}\right)$$

Finally, denote by n_i the length of the sequence t_{α_i}.

The main application of nice sequences is to construct functions representing the cardinals in the sequences $\vec{\mu}^*_{\alpha_i}$ in $\text{Ult}\left(V\left[G\right], W\right)$, using only functions in V and partial information about Prikry sequences added in $V\left[G\right]$. We demonstrate this, working by induction.

1. For every $\xi < \kappa$ with $\left|d^{-1}(\xi)\right| = m$, recall the sequence $\langle \mu^{*1}_0(\xi), \ldots, \mu^{*m}_0(\xi)\rangle$ which is the increasing enumeration of $d^{-1}(\xi)$. Denote–

$$\vec{\mu}^*_0(\xi) = \langle \xi, \mu^{*1}_0(\xi), \ldots, \mu^{*m}_0(\xi)\rangle$$

   Then in $M\left[H\right]$, the sequence $\left[\xi \mapsto \vec{\mu}^*_0(\xi)\right]_W$ is–

$$\langle \kappa = \mu_0, \mu^{*1}_0, \ldots, \mu^{*m-1}_0, \mu^{*m}_0\rangle = \langle \left[Id\right]_W, \left[Id\right]_{W^1}, \ldots, \left[Id\right]_{W^{m-1}}, \left[Id\right]_{W^m}\rangle$$

2. Given $\xi < \kappa$, let $\mu_{\alpha_1}(\xi)$ be the (n_1+1)-th element in the Prikry sequence of $g_1 = g_1(\vec{\mu}^*_0(\xi))$ (typically, this is the element which appears in this sequence after the initial segment $f_1\left(\vec{\mu}^*_0(\xi)\right)$, which represents t_{α_1} in $\text{Ult}\left(V\left[G\right], W\right)$). Let $\langle \mu^{*1}_{\alpha_1}(\xi), \ldots, \mu^{*m_{\alpha_1}-1}_{\alpha_1}(\xi)\rangle$ be the increasing enumeration of $d^{-1}\left(\mu_{\alpha_1}(\xi)\right)$ below g_1. Denote $\mu^{*m_{\alpha_1}}_{\alpha_1}(\xi) = g_1$ and–

$$\vec{\mu}^*_{\alpha_1}(\xi) = \langle \mu_{\alpha_1}(\xi), \mu^{*1}_{\alpha_1}(\xi), \ldots, \mu^{*m_1-1}_{\alpha_1}(\xi)\rangle$$

   Then in $M\left[H\right] \simeq \text{Ult}\left(V\left[G\right], W\right)$,

$$\left[\xi \mapsto \vec{\mu}^*_{\alpha_1}(\xi)\right]_W = \langle \mu_{\alpha_1}, \mu^{*1}_{\alpha_1}, \ldots, \mu^{*m_1-1}_{\alpha_1}\rangle$$

   namely, this sequence starts with μ_{α_1}, concatenated with the increasing enumeration of $d^{-1}\{\mu_{\alpha_1}\}$ in $M\left[H\right]$. Let us verify this. Assume for simplicity that t_{α_1} is empty, namely $n_1 = 0$, or, in other words, μ_{α_1} is the first element in the Prikry sequence of $k_{\alpha_1}\left(\mu_{\alpha_1}\right)$ (the fact that μ_{α_1} appears in this Prikry sequence, follows from property (C) of k_{α_1}). The first element in $\left[\xi \mapsto \vec{\mu}^*_{\alpha_1}(\xi)\right]_W$ is–

$$d\left(\left[\xi \mapsto g_1\left(\vec{\mu}^*_0(\xi)\right)\right]_W\right) = d\left(j_W\left(g_1\right)\left(\left[Id\right]_W, \left[Id\right]_{W^1}, \ldots, \left[Id\right]_{W^m}\right)\right) = d\left(k_{\alpha_1}\left(\mu_{\alpha_1}\right)\right) = \mu_{\alpha_1}$$

   From this it is implied that the rest of the elements in $\left[\xi \mapsto \vec{\mu}^*_{\alpha_1}(\xi)\right]_W$ are the increasing enumeration of $d^{-1}\{\mu_{\alpha_1}\}$ below $k_{\alpha_1}\left(\mu_{\alpha_1}\right)$, which is exactly $\langle \mu^{*1}_{\alpha_1}, \ldots, \mu^{*m_{\alpha_1}-1}_{\alpha_1}\rangle$.

3. Assuming that $0 \le j < k$ and the functions $\vec{\mu}^*_{\alpha_0}(\xi), \ldots, \vec{\mu}^*_{\alpha_j}(\xi)$ have been defined, let $\mu_{\alpha_{j+1}}(\xi)$ be the n_{j+1}-th element in the Prikry sequence of $g_{j+1}\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_j}(\xi)\right)$ in $M\left[H\right]$.

Let $\langle \mu^{*1}_{\alpha_{j+1}}(\xi), \ldots, \mu^{*m_{j+1}-1}_{\alpha_{j+1}}(\xi) \rangle$ be the increasing enumeration of $d^{-1}\left(\mu_{\alpha_{j+1}}(\xi)\right)$ below–

$$\mu^{*m_{j+1}}_{\alpha_{j+1}}(\xi) = g_{j+1}\left(\vec{\xi}, \vec{\mu}^{*}_{\alpha_0}(\xi), \ldots, \vec{\mu}^{*}_{\alpha_j}(\xi)\right)$$

Also, denote–

$$\vec{\mu}^{*}_{\alpha_{j+1}}(\xi) = \langle \mu^{*1}_{\alpha_{j+1}}(\xi), \ldots, \mu^{*m_{j+1}-1}_{\alpha_{j+1}}(\xi) \rangle$$

Then in $M[H] \simeq \mathrm{Ult}(V[G], W)$,

$$\left[\xi \mapsto \vec{\mu}^{*}_{\alpha_{j+1}}(\xi)\right]_W = \langle \mu_{\alpha_{j+1}}, \mu^{*1}_{\alpha_{j+1}}, \ldots, \mu^{*m_{j+1}-1}_{\alpha_{j+1}} \rangle$$

namely, this sequence starts with $\mu_{\alpha_{j+1}}$, concatenated with the increasing enumeration of $d^{-1}\{\mu_{\alpha_{j+1}}\}$ in $M[H]$. This is proved similarly to the previous point.

Denote $\mu_\alpha = \mathrm{crit}(k_\alpha)$. Write $\mu_\alpha = j_\alpha(h)(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k})$, where $\alpha_0 < \ldots < \alpha_k < \alpha$ is a nice sequence. Let $m_0, \ldots, m_k$ be such that $m_i = m_{\alpha_i}$. Denote $m = m_0$. Let g_i, f_i, F^j_i be functions as above.

Note that, by induction, $\left[\xi \mapsto F^j_{i+1}(\xi, \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_i}(\xi))\right]_W = \left[\xi \mapsto U^j_{\mu_{\alpha_{i+1}}(\xi)}\right]_W$ for every $0 \leq i \leq k$ and $0 \leq j \leq m_{\alpha_{i+1}}$ (Recall that, for a measurable $\eta \in \Delta$, U^j_η is the j-th measure in the system $U^0_\eta \lhd \ldots \lhd U^{m_\eta}_\eta$ associated with η). Thus, for a set of ξ-s in W,

$$F^j_{i+1}(\xi, \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_i}(\xi)) = U^j_{\mu_{\alpha_{i+1}}(\xi)}$$

Definition 3.4.12. *Fix a nice sequence $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k$ below α. Given a condition $p \in P_\kappa$ and a sequence of increasing sequences–*

$$\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle = \langle \langle \xi, \xi^1, \ldots, \xi^m \rangle, \langle \nu_1, \nu^1_1, \ldots, \nu^{m_0-1}_1 \rangle, \langle \nu_2, \nu^1_2, \ldots, \nu^{m_1-1}_2 \rangle, \ldots\ldots, \langle \nu_k, \nu^1_k, \ldots, \nu^{m_k-1}_k \rangle \rangle$$

we define whenever $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle$ is admissible for p, and in this case, we define an extension $p^\frown \langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \geq p$.

1. *An increasing sequence $\vec{\xi} = \langle \xi, \xi^1, \ldots, \xi^m \rangle$ is admissible for p if it is admissible for p in the sense of definition 3.4.9. If it is, the extension $p^\frown \langle \xi, \xi^1, \ldots, \xi^m \rangle$ is defined the same as in 3.4.9.*

2. *Let $1 \leq i < k$. Assume that $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i \rangle$ is admissible for p and $q = p^\frown \langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i \rangle$ has been defined. Denote–*

$$g_{i+1} = g_{i+1}\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\right)$$

$$t_{i+1} = f_{i+1}\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\right)$$

$$F^j_{i+1} = F^j_{i+1}\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\right) \ \ (0 \leq j < m_{\alpha_{i+1}})$$

We say that $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_{i+1}\rangle$ *is admissible for* p *if, in the forcing* $P\restriction_{g_{i+1}}$*, the sequence* $\vec{\nu}_{i+1} = \langle \nu_{i+1}, \nu^1_{i+1}, \ldots, \nu^{m_{i+1}-1}_{i+1}\rangle$ *is admissible for* $q\restriction_{g_{i+1}}$ *in the sense of definition 3.4.9, and if–*

$$\left(q\restriction_{g_{i+1}}\right)^\frown \langle \vec{\nu}_{i+1}\rangle \Vdash \langle t_{i+1}{}^\frown\langle \nu_{i+1}\rangle, \underset{\sim}{A}^q_{g_{i+1}}\rangle \text{ extends } q\left(g_{i+1}\right), \text{ and } \langle F^j_{i+1} : j \leq m_{\alpha_{i+1}}\rangle$$
$$\text{is the system of measures } \langle U^j_{g_{i+1}} : j \leq m\left(U^\times_{g_{i+1}}\right)\rangle.$$

Assuming this holds, let–

$$p^\frown\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_{i+1}\rangle = \left(\left(q\restriction_{g_{i+1}}\right)^\frown \langle \vec{\nu}_{i+1}\rangle\right)^\frown \langle t_{i+1}{}^\frown\langle \nu_{i+1}\rangle, \underset{\sim}{A}^q_{g_{i+1}}\rangle^\frown q \setminus (g_{i+1}+1)$$

Given $i < \omega$ *and a condition* p *which forces that–*

$$\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\rangle = \langle \vec{\mu}^*_0(\xi), \vec{\mu}^*_1(\xi), \ldots, \vec{\mu}^*_i(\xi)\rangle$$

We define, similarly to above, whenever a sequence

$$\langle\langle \nu_{i+1}, \nu^1_{i+1}, \ldots, \nu^{m_{i+1}-1}_{i+1}\rangle, \ldots\ldots, \langle \nu_k, \nu^1_k, \ldots, \nu^{m_k-1}_k\rangle\rangle$$

is admissible for p *above* $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\rangle$*. If this is the case, we can define similarly the condition* $p^\frown\langle\langle \nu_{i+1}, \nu^1_{i+1}, \ldots, \nu^{m_{i+1}-1}_{i+1}\rangle, \ldots\ldots, \langle \nu_k, \nu^1_k, \ldots, \nu^{m_k-1}_k\rangle\rangle$.

Theorem 3.4.13 (Multivariable Fusion)**.** *Let* $p \in P_\kappa$ *be a condition and, for every sequence–*

$$\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle = \langle\langle \xi, \xi^1, \ldots, \xi^m\rangle, \langle \nu_1, \nu^1_1, \ldots, \nu^{m_1-1}_1\rangle, \ldots, \langle \nu_k, \nu^1_k, \ldots, \nu^{m_k-1}_k\rangle\rangle$$

let $e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ *be a* P_{ν_k}*-name for a subset of* $P \setminus \nu_k$ *which is* $\leq^*$ *dense open above any condition which forces that* $\langle \vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\rangle = \langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle$*. Then there exists* $p^* \geq^* p$ *and a set* $X \in \mathcal{E}^{'}_0$*, such that for every sequence of increasing sequences,*

$$\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle = \langle\langle \xi, \xi^1, \ldots, \xi^m\rangle, \langle \nu_1, \nu^1_1, \ldots, \nu^{m_1-1}_1\rangle, \ldots, \langle \nu_k, \nu^1_k, \ldots, \nu^{m_k-1}_k\rangle\rangle$$

which is admissible for p^**, and such that* $\langle \xi, \xi^1, \ldots, \xi^m\rangle \in X$*,*

$$p^{*\frown}\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle\restriction_{\nu_k} \Vdash p^{*\frown}\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle \setminus \nu_k \in e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$$

Furthermore, there exists $p^* \in G$*, for which–*

$$\{\xi < \kappa \colon \langle \vec{\xi}, \vec{\mu}_{\alpha_1}(\xi), \dots, \vec{\mu}_{\alpha_k}(\xi) \rangle \textit{ is admissible for } p^*,$$
$$p^{*\frown}\langle \vec{\xi}, \vec{\mu}^*_{\alpha_1}(\xi), \dots, \vec{\mu}^*_{\alpha_k}(\xi) \rangle \in G \textit{ and}$$
$$p^{*\frown}\langle \vec{\xi}, \vec{\mu}^*_{\alpha_1}, \dots, \vec{\mu}^*_{\alpha_k} \rangle \restriction_{\mu_{\alpha_k}} \Vdash p^{*\frown}\langle \vec{\xi}, \vec{\mu}^*_{\alpha_1}, \dots, \vec{\mu}^*_{\alpha_k} \rangle \setminus \mu_{\alpha_k} \in e\left(\vec{\xi}, \vec{\mu}^*_{\alpha_1}, \dots, \vec{\mu}^*_{\alpha_k}\right)\} \in W$$

Proof. For every $1 \le i \le k$ and a sequence $\langle \vec{\xi}, \dots, \vec{\nu}_i \rangle$, we define a set $e\left(\vec{\xi}, \vec{\nu}_1, \dots, \vec{\nu}_i\right)$, which is $\le^*$ dense open above conditions which force that–

$$\langle d^{-1}\{\xi\}, \langle \mu_{\alpha_1}(\xi), \mu^{*1}_{\alpha_1}(\xi), \dots, \mu^{*m_1-1}_{\alpha_1}(\xi) \rangle, \dots, \langle \mu_{\alpha_i}(\xi), \mu^{*1}_{\alpha_i}(\xi), \dots, \mu^{*m_i-1}_{\alpha_i}(\xi) \rangle \rangle =$$
$$\langle \langle \xi^1, \dots, \xi^m \rangle, \langle \nu_1, \nu_1^1, \dots, \nu_1^{m_1-1} \rangle, \dots, \langle \nu_i, \nu_i^1, \dots, \nu_i^{m_i-1} \rangle \rangle$$

as follows:

$$e\left(\vec{\xi}, \vec{\nu}_1, \dots, \vec{\nu}_i\right) = \{r \in P \setminus \nu_i \colon \text{ for every } \langle \vec{\nu}_{i+1}, \dots, \vec{\nu}_k \rangle \text{ which is admissible for}$$
$$r \text{ above } \langle \vec{\xi}, \vec{\nu}_1, \dots, \vec{\nu}_i \rangle, r^\frown \langle \vec{\nu}_{i+1}, \dots, \vec{\nu}_k \rangle \restriction_{\nu_k} \Vdash$$
$$r^\frown \langle \vec{\nu}_{i+1}, \dots, \vec{\nu}_k \rangle \setminus \nu_k \in e\left(\vec{\xi}, \vec{\nu}_1, \dots, \vec{\nu}_k\right)\}$$

Lemma 3.4.14. *If* $e\left(\vec{\xi}, \vec{\nu}_1, \dots, \vec{\nu}_i, \vec{\nu}_{i+1}\right)$ *is* $\le^*$*-dense open above conditions which force that–*

$$\langle \vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \dots, \vec{\mu}^*_{\alpha_{i+1}}(\xi) \rangle = \langle \vec{\xi}, \vec{\nu}_1, \dots, \vec{\nu}_i, \vec{\nu}_{i+1} \rangle$$

then $e\left(\vec{\xi}, \vec{\nu}_1, \dots, \vec{\nu}_i\right)$ *is* $\le^*$*-dense open above conditions which force that–*

$$\langle \vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \dots, \vec{\mu}^*_{\alpha_i}(\xi) \rangle = \langle \vec{\xi}, \vec{\nu}_1, \dots, \vec{\nu}_i \rangle$$

Proof. Fix $\langle \vec{\xi}, \vec{\nu}_1, \dots, \vec{\nu}_i \rangle$. Let $r \in P \setminus \nu_i$ be a condition which forces that–

$$\langle \vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \dots, \vec{\mu}^*_{\alpha_i}(\xi) \rangle = \langle \vec{\xi}, \vec{\nu}_1, \dots, \vec{\nu}_i \rangle$$

Denote for simplicity $m = m_{i+1}$ and–

$$g_{i+1} = g_{i+1}\left(\vec{\xi}, \vec{\nu}_1, \dots, \vec{\nu}_i\right) \ , \ t_{i+1} = f_{i+1}\left(\vec{\xi}, \vec{\nu}_1, \dots, \vec{\nu}_i\right) \ , \ F^j_{i+1} = F^j_{i+1}\left(\vec{\xi}, \vec{\nu}_1, \dots, \vec{\nu}_i\right) \ (0 \le j < m)$$

We apply theorem 3.4.10. For that, consider the forcing $P \restriction_{(\nu_i, g_{i+1})}$ and the sequence $F^0_{i+1} \lhd F^1_{i+1} \lhd \dots \lhd F^{m-1}_{i+1}$ of measures on g_{i+1}. We describe a set $d(\nu_{i+1}, \nu^1_{i+1}, \dots, \nu^{m-1}_{i+1}) \subseteq P \restriction_{(\nu_i, g_{i+1})} \setminus \nu_{i+1}$

which is $\leq^*$ dense open above conditions which force that $d^{-1}\{\nu_{i+1}\} = \langle \nu^1_{i+1}, \ldots, \nu^{m-1}_{i+1}\rangle$:

$$d(\nu_{i+1}, \nu^1_{i+1}, \ldots, \nu^{m-1}_{i+1}) = \{s \in P \restriction_{(\nu_{i+1}, g_{i+1})} : \text{ if } s \Vdash \langle t_{i+1} {}^\frown \langle \nu_{i+1}\rangle, \underset{\sim}{A}^r_{g_{i+1}}\rangle \geq r(g_{i+1}), \text{ then there exists a direct extension } q \geq^* \langle t_{i+1} {}^\frown \langle \nu_{i+1}\rangle, \underset{\sim}{A}^r_{g_{i+1}}\rangle^\frown r \setminus (g_{i+1}+1) \text{ such that } s^\frown q \in e\left(\vec{\xi}, \ldots, \vec{\nu}_i, \vec{\nu}_{i+1}\right)\}$$

By theorem 3.4.10, there exists $r^* \restriction_{g_{i+1}} \geq^* r \restriction_{g_{i+1}}$ and a set X which belongs to the product measure $\mathcal{E}_{i+1} = F^0_{i+1} \times F^1_{i+1} \times \ldots \times F^{m-1}_{i+1}$, such that for every increasing $\langle \nu_{i+1}, \nu^1_{i+1}, \ldots, \nu^{m-1}_{i+1}\rangle \in X$,

$$r^* \restriction_{\nu_{i+1}} \Vdash r^{*\frown} \langle \nu_{i+1}, \nu^1_{i+1}, \ldots, \nu^{m-1}_{i+1}\rangle \setminus \nu_{i+1} \in d\left(\nu_{i+1}, \nu^1_{i+1}, \ldots, \nu^{m-1}_{i+1}\right)$$

Let us define $r^* \setminus g_{i+1}$. Assume that we work in the generic extension for $P_{g_{i+1}}$, and $r^* \restriction_{g_{i+1}}$, which is already defined, belongs to it. For every $\nu_{i+1} \in A^r_{g_{i+1}}$ above $\max t_{i+1}$, we denote $d^{-1}(\nu_{i+1}) = \langle \nu^1_{i+1}, \ldots, \nu^{m-1}_{i+1}\rangle$. Let $q(\nu_{i+1}) \geq^* \langle t_{i+1} {}^\frown \langle \nu_{i+1}\rangle, A^r_{g_{i+1}}\rangle^\frown r \setminus g_{i+1}$ be a condition such that–

$$\left(r^{*\frown}\langle \nu_{i+1}, \ldots, \nu^{m-1}_{i+1}\rangle \setminus \nu_{i+1}\right)^\frown q(\nu_{i+1}) \in e\left(\vec{\xi}, \ldots, \vec{\nu}_{i+1}\right)$$

(and take $q(\nu_{i+1}) = r \setminus g_{i+1}$ if such q does not exist).

We can now define $r^*(g_{i+1})$. Generate from the set $X \in \mathcal{E}_{i+1}$ a corresponding set $Y \in W_{g_{i+1}}$. Just pick Y to be a set such that every increasing sequence from $Y \times \pi^{-1}_{1,0} Y \times \ldots \times \pi^{-1}_{m-1,0} Y$ belongs to X. Let–

$$r^*(g_{i+1}) = \langle \langle\rangle, A^r_{g_{i+1}} \cap (Y \cup \max t_{i+1}) \cap \left(\left(\triangle_{\nu_{i+1} < g_{i+1}} A^{q(\nu_{i+1})}_{g_{i+1}}\right) \cup \max t_{i+1}\right)\rangle$$

Finally, let us define $r^* \setminus (g_{i+1}+1)$ to be $q(\underset{\sim}{\nu}_{i+1}) \setminus (g_{i+1}+1)$, where here, $\underset{\sim}{\nu}_{i+1} = d(g_{i+1})$ can be read from the generic up to $g_{i+1}+1$. This concludes the definition of $r^* \in e\left(\vec{\xi}, \ldots, \vec{\nu}_i\right)$. □

Inductively, it follows that for every increasing sequence $\vec{\xi} = \langle \xi, \xi^1, \ldots, \xi^m\rangle$, the set $e(\vec{\xi})$, defined similarly as above, is $\leq^*$-dense open above conditions which force that $d^{-1}\{\xi\} = \langle \xi^1, \ldots, \xi^m\rangle$. Apply theorem 3.4.10 one more time to obtain, from the condition p given in the formulation of the theorem, the required direct extension p^*. □

3.4.4 Proof of Properties (B)-(E)

Lemma 3.4.15. *$\mu_\alpha = crit(k_\alpha)$ is measurable in M_α.*

Proof. Write $\mu_\alpha = j_\alpha(h)\left(j_{1,\alpha}\left([Id]_0'\right), j_{\alpha_1+1,\alpha}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right)$. We can assume that for every $\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k$,

$$h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) > \nu_k$$

since $\mu_\alpha > \mu_{\alpha_k}$ (this can be done by modifying the value of $h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ to 0 whenever $h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \leq \nu_k$). We can also assume similarly that $h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ is a regular cardinal.

Let $f \in V[G]$ be a function such that $\mu_\alpha = [f]_W$. Let us assume, for contradiction, that for every $\langle\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle$, $h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ is non-measurable.

By changing f on a set outside of W, we can also assume that for every $\xi < \kappa$,

$$f(\xi) < h\left(\vec{\xi}, \vec{\mu}_{\alpha_1}(\xi), \ldots, \vec{\mu}_{\alpha_k}(\xi)\right) \tag{3.5}$$

Indeed, this can be done since–

$$[f]_W = \mu_\alpha < k_\alpha(\mu_\alpha) = \left[\xi \mapsto h\left(\vec{\xi}, \vec{\mu}_{\alpha_1}(\xi), \ldots, \vec{\mu}_{\alpha_k}(\xi)\right)\right]_W$$

Let $p \in G$ be a condition which forces that for every $\xi < \kappa$, equation 3.5 holds. From now on, work with conditions in $P = P_\kappa$ above p. We define, for every–

$$\langle\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle = \langle\langle\xi, \xi^1, \ldots, \xi^m\rangle, \langle\nu_1, \nu_1^1, \ldots, \nu_1^{m_1-1}\rangle, \ldots, \langle\nu_k, \nu_k^1, \ldots, \nu_k^{m_k-1}\rangle\rangle$$

a set–

$$e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \subseteq P \setminus \nu_k$$

which is $\leq^*$ dense open above conditions which force that $\langle\vec{\mu}_{\alpha_1}(\xi), \ldots, \vec{\mu}_{\alpha_k}(\xi)\rangle = \langle\vec{\nu}_1, \ldots, \vec{\nu}_k\rangle$:

$$e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) = \{r \in P \setminus \nu_k \colon \exists\alpha < h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right),\ r \Vdash \underset{\sim}{f}(\xi) < \check{\alpha}\}$$

The $\leq^*$-density of $e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ essentially uses the fact that $h = h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ is not measurable; employing this, $\underset{\sim}{f}(\xi)$ can be reduced to a P_h-name, and thus be evaluated by less then h many possibilities by lemma 3.2.8.

Let us apply now the Multivariable Fusion Lemma. There exists $p^* \in G$ and sets $X_0 \in U^0, \ldots, X_m \in U^m$ such that for every sequence of sequences $\langle\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle$ which is admissible for p^*,

$$\left(p^{*\frown}\langle\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle\right)\restriction_{\nu_k} \Vdash \left(p^{*\frown}\langle\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle\right) \setminus \nu_k \in e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$$

whenever $\vec{\xi} \in X_0 \times \ldots X_m$ is increasing.

For every such $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle$, let–

$$A\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) = \{\gamma < h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) : \exists q \geq \left(p^{*\frown}\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle\right) \restriction_{\nu_k}, \\ q \Vdash \gamma = \underset{\sim}{\alpha}\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)\}$$

where $\underset{\sim}{\alpha}\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ is the P_{ν_k}-name for the ordinal α which witnesses the fact that $\left(p^{*\frown}\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle\right) \setminus \nu_k \in e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$. Then $A\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ is a bounded subset of $h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$, since $\nu_k < h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ and $\mathrm{GCH}_{\leq\kappa}$ holds in V.

For a set of ξ-s in W,

$$p^* \Vdash \underset{\sim}{f}(\xi) \in A\left(\vec{\xi}, \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$$

In particular, in $M[H]$,

$$[f]_W \in j_W\left(\langle \vec{\xi}, \vec{\nu}_0, \ldots, \vec{\nu}_k \rangle \mapsto A\left(\vec{\xi}, \vec{\nu}_0, \ldots, \vec{\nu}_k\right)\right)\left(\vec{\kappa}^*, \vec{\mu}^*_{\alpha_1}, \ldots, \vec{\mu}^*_{\alpha_k}\right) = \\ k_\alpha\left(j_\alpha\left(\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \mapsto A\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)\right)\left(j_{1,\alpha}\left([Id]_0^{'}\right), j_{\alpha_1+1,\alpha}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right)\right)$$

But–

$$\left| j_\alpha\left(\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \mapsto A\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)\right)\left(j_{1,\alpha}\left([Id]_0^{'}\right), j_{\alpha_1+1,\alpha}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right)\right| < \\ j_\alpha(h)\left(j_{1,\alpha}\left([Id]_0^{'}\right), j_{\alpha_1+1,\alpha}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right) = \mu_\alpha$$

and thus $\mu_\alpha = [f]_W \in \mathrm{Im}(k_\alpha)$, a contradiction. □

Corollary 3.4.16. *μ_α is the least measurable μ in M_α such that $\mu \geq \sup\{\mu_{\alpha'} : \alpha' < \alpha\}$ and $(cf(\mu))^V > \kappa$.*

Proof. Assume for contradiction that there exists a measurable λ in M_α, such that $\sup\{\mu_{\alpha'} : \alpha' < \alpha\} \leq \lambda < \mu_\alpha$. Let us argue that $(\mathrm{cf}(\lambda))^V \leq \kappa$.

First, $\lambda = k_\alpha(\lambda)$, since $\mathrm{crit}(k_\alpha) = \mu_\alpha$, and thus λ is measurable in M. Note that $\lambda < \mu_\alpha < \kappa^* = j_\alpha(\kappa)$, so $\lambda < j_W(\kappa)$ and thus λ has cofinality ω in $M[H]$. Thus, $(\mathrm{cf}(\lambda))^{V[G]} = \omega$, and, in particular, $(\mathrm{cf}(\lambda))^V < \kappa$. □

Lemma 3.4.17. *μ_α appears as an element in the Prikry sequence of $k_\alpha(\mu_\alpha)$.*

Proof. In $M[H]$, denote by t^* the initial segment of the Prikry sequence of $k_\alpha(\mu_\alpha)$ which consists of all the ordinals below μ_α. Denote by n^* the length of t^*. Let $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \mapsto t^*\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$

be a function in V such that–

$$t^* = j_\alpha\left(\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle \mapsto t^*\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)\right)\left(j_{1,\alpha}\left([Id]^{'}_{\alpha}\right), j_{\alpha_1+1,\alpha}\left([Id]_{\alpha_1}\right),\ldots,j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right)$$

(this can be done by modifying the nice sequence $\langle\vec{\alpha}_1,\ldots,\vec{\alpha}_k\rangle$, if necessary, so that t^* can be represented by it). We can assume that for every $\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle$, $t^*\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$ is a sequence of length n^*. Since $k_\alpha\left(t^*\right) = t^*$,

$$\left[\xi \mapsto t^*\left(\vec{\xi},\vec{\mu}^*_{\alpha_1}(\xi),\ldots,\vec{\mu}^*_{\alpha_k}(\xi)\right)\right]_W = t^*$$

In $V[G]$, denote, for every $\xi < \kappa$,

$$\mu_\alpha(\xi) = \text{ the } \left(n^*+1\right)\text{-th element in the Prikry sequence of } h\left(\vec{\xi},\vec{\mu}^*_{\alpha_1}(\xi),\ldots,\vec{\mu}^*_{\alpha_k}(\xi)\right)$$

Clearly $[\xi \mapsto \mu_\alpha(\xi)]_W \geq \mu_\alpha$. We argue that equality holds. We will prove that for every $\eta < [\xi \mapsto \mu_\alpha(\xi)]_W$, $\eta < \mu_\alpha$. Assume that such η is given, and let $f \in V[G]$ be a function such that $[f]_W = \eta$. Then we can assume that for every $\xi < \kappa$,

$$f(\xi) < \mu_\alpha(\xi)$$

and let $p \in G$ be a condition which forces this.

Let us apply now the Multivariable Fusion Lemma. For every $\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle$, consider the set–

$$\begin{aligned} e\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right) = \{r \in P\setminus\nu_k \colon \exists\alpha < h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right),\ r \Vdash \text{ if } t^*\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right) \\ \text{is an initial segment of the Prikry sequence of } h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right), \\ \text{then } \underset{\sim}{f}(\xi) < \alpha\} \end{aligned}$$

then $e\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$ is $\leq^*$ dense open above conditions which force that $d^{-1}\{\xi\} = \langle\xi^1,\ldots,\xi^m\rangle$ and $\langle\vec{\nu}_0,\ldots,\vec{\nu}_k\rangle = \langle\vec{\mu}_{\alpha_0}(\xi),\ldots,\vec{\mu}_{\alpha_k}(\xi)\rangle$. This follows in several steps: First, use the $\leq^*$-closure to reduce $\underset{\sim}{f}(\xi)$ to a P_{h+1}-name, where $h = h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$. This can be done by taking a direct extension of a given condition in the forcing $P\setminus(h+1)$. Second, reduce $\underset{\sim}{f}(\xi)$ to a P_h name, by applying on the Prikry forcing at coordinate h the following fact: If $t^* = t^*\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$ is an initial segment of the Prikry sequence of h, and a given an ordinal is forced to be below the successor of t^* in this sequence, then its value can be decided by taking a direct extension. Finally, apply lemma 3.2.10 and direct extend in the forcing $P_{(\nu_k,h)}$, to bound the value of $\underset{\sim}{f}(\xi)$ by an ordinal below h.

Thus, there exists $p^* \in G$, such that for every $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle$ which is admissible for p^*,

$$p^{*\frown}\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \restriction_{\nu_k} \Vdash p^{*\frown}\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \setminus \nu_k \in e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$$

in particular, $p^{*\frown}\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \restriction_{\nu_k}$ forces that there exists $\alpha < h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ such that–

$$p^{*\frown}\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \setminus \nu_k \Vdash \underset{\sim}{f}(\xi) < \check{\alpha} \tag{3.6}$$

Let–

$$A\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) = \{\gamma \colon \exists q \geq p^{*\frown}\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \restriction_{\nu_k},\ q \Vdash \underset{\sim}{\alpha} = \gamma\}$$

(where, as in the previous lemma, $\underset{\sim}{\alpha}$ is a P_{ν_k}-name for the ordinal α in equation 3.6).
Then $p^* \Vdash \underset{\sim}{f}(\xi) \in A\left(\vec{\xi}, \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$, and $A\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ is a bounded subset of $h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$. Arguing as lemma 3.4.15, it follows that $\eta = [f]_W \in \operatorname{Im}(k_\alpha) \cap k_\alpha(\mu_\alpha) = \mu_\alpha$, as desired. □

Lemma 3.4.18. *Assume that for every measurable $\xi < \kappa$, $\underset{\sim}{U}_\xi$ is a P_ξ-name for a measure on ξ which belongs to V. Let $\underset{\sim}{U} = j_\alpha(\xi \mapsto \underset{\sim}{U}_\xi)(\mu_\alpha)$. Then there exists a set $\mathcal{F} \in M_\alpha$ of measures on μ_α in M_α, with $|\mathcal{F}| < \mu_\alpha$, such that, for some $p \in G$,*

$$(j_W(p))^\frown \langle \vec{\mu}^*_{\alpha_0}, \ldots, \vec{\mu}^*_{\alpha_k} \rangle \Vdash k_\alpha(\underset{\sim}{U}) \in k''_\alpha \mathcal{F}$$

In particular, there exists a measure $F \in \mathcal{F}$ such that $k_\alpha(F) = (k_\alpha(\underset{\sim}{U}))_H$.

Proof. For every $\xi < \kappa$, fix an enumeration s_ξ of all the normal measures on ξ in V. Apply Multivariable Fusion. Define for every $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle$ the set–

$$\begin{aligned} e\left(\vec{\xi}, \vec{\nu}_0, \ldots, \vec{\nu}_k\right) = \{ & r \in P \setminus \nu_k \colon \text{there exists a set of ordinals } A \text{ of cardinality} \\ & \text{strictly smaller than } h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right),\ \text{such that} \\ & r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \Vdash \underset{\sim}{U}_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \in s''_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} A\} \end{aligned}$$

As before, there exists $p^* \geq^* p$ in G, such that for every $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle$ which is admissible for p^*,

$$p^{*\frown}\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \restriction_{\nu_k} \Vdash p^{*\frown}\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \setminus \nu_k \in e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$$

Let $A\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ be the set of ordinals, forced by some extension of $p^{*\frown}\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \restriction_{\nu_k}$, to be an element of the set $\underset{\sim}{A}$ above. Then $A\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ is a set of ordinals of cardinality strictly below $h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$.

In $V[G]$,

$$\{\xi<\kappa\colon p^{*\frown}\langle\vec{\xi},\vec{\mu}^{*}_{\alpha_1}(\xi),\ldots,\vec{\mu}^{*}_{\alpha_k}(\xi)\rangle\Vdash U_{h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)}\in s''_{h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)}A^{*}\left(\vec{\xi},\vec{\mu}^{*}_{\alpha_1}(\xi),\ldots,\mu^{*}_{\alpha_k}(\xi)\right)\}\in W$$

Now, in M_α, denote–

$$\mathcal{F}=(j_\alpha(s)_{\mu_\alpha})''\,j_\alpha\left(\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle\mapsto A^{*}\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)\right)\left(j_{1,\alpha}\left([Id]'_0\right),j_{\alpha_1+1,\alpha}\left([Id]_1\right),\ldots,j_{\alpha_k+1,\alpha}\left([Id]_k\right)\right)$$

Then $|\mathcal{F}|<\mu_\alpha$, and, in $M[H]$,

$$(j_W(p^*))^{\frown}\langle\vec{\mu}^{*}_{\alpha_0},\ldots,\vec{\mu}^{*}_{\alpha_k}\rangle\Vdash k_\alpha\left(\underset{\sim}{\mathcal{U}}\right)\in k''_\alpha\mathcal{F}$$

□

We now apply the above lemma on specific names for measures $\underset{\sim}{U}_\xi$. For every measurable $\xi<\kappa$, let $W_\xi=U^{\times}_\xi$ be the measure used to singularize ξ at stage ξ in the iteration $P=P_\kappa$. Each such $W^{\times}_\xi$ can be assigned to a sequence of Rudin-Keisler equivalent measures on ξ,

$$W^0_\xi,\ldots,W^{m_\xi}_\xi=W_\xi$$

as defined in section 2. Denote $U^j_\xi=W^j_\xi\cap V\in V$ for every $0\le j\le m_\xi$. Then–

$$U^0_\xi\lhd U^1_\xi\lhd\ldots\lhd U^{m_\xi}_\xi$$

Corollary 3.4.19. *For every $1\le j<m_\alpha$, there exists a measure $U^j_{\mu_\alpha}\in M_\alpha$ on μ_α, such that–*

$$k_\alpha\left(U^j_{\mu_\alpha}\right)=j_W\left(\xi\mapsto U^j_\xi\right)(k_\alpha(\mu_\alpha))=\left[\xi\mapsto U^j_{h\left(\vec{\mu}^{*}_{\alpha_0}(\xi),\ldots,\vec{\mu}^{*}_{\alpha_k}(\xi)\right)}\right]_W$$

We consider the above corollary as the definition of the measures $U^j_{\mu_\alpha}$ for every $1\le j<m_\alpha$. Note that, by elementarity, $U^0_{\mu_\alpha}\lhd U^1_{\mu_\alpha}\lhd\ldots\lhd U^{m_\alpha-1}_{\mu_\alpha}$. Let $\mathcal{E}_\alpha$ be the measure on $[\kappa]^{m_\alpha}$ which corresponds to the iterated ultrapower with $U^{m_\alpha-1}_{\mu_\alpha}\rhd\ldots\rhd U^0_{\mu_\alpha}$ in decreasing order. We argue that $\mathcal{E}_\alpha$ is derived from $k_\alpha\colon M_\alpha\to M$.

Lemma 3.4.20. $U^0_{\mu_\alpha}=\{X\subseteq\mu_\alpha\colon\mu_\alpha\in k_\alpha(X)\}\cap M_\alpha$. *Furthermore, if $m_\alpha>1$, then for every $1\le j\le m_\alpha-1$,*

$$U^j_{\mu_\alpha}=\{X\subseteq\mu_\alpha\colon\mu^{*j}_\alpha\in k_\alpha(X)\}\cap M_\alpha$$

Remark 3.4.21.

1. *Let us note that for every j as above, $\mu_\alpha < \mu_\alpha^j < k_\alpha(\mu_\alpha)$, so $U^{*j}_{\mu_\alpha}$ is an ultrafilter which concentrates on μ_α.*

2. *We deliberately did not define, in corollary 3.4.19, the measure $U^{m_\alpha}_{\mu_\alpha}$ - it is not derived from k_α and does not participate in $j_W \restriction_V$. The exception is $\alpha = 0$ where $U^m = U$ is the first step in the iteration.*

Proof. We first provide the proof for $U^0_{\mu_\alpha}$. Assume that $X \in M_\alpha$ and $\mu_\alpha \in k_\alpha(X)$. Write–

$$X = j_\alpha \left(\langle \vec{\xi}, \vec{\nu}_0, \ldots, \vec{\nu}_k \rangle \mapsto X\left(\vec{\xi}, \vec{\nu}_0, \ldots, \vec{\nu}_k\right)\right) \left(j_{1,\alpha}\left(\vec{\kappa}\right), j_{\alpha_0,\alpha}\left(\vec{\mu}_{\alpha_0}\right), \ldots, j_{\alpha_k,\alpha}\left(\vec{\mu}_{\alpha_k}\right)\right)$$

where, without loss of generality, the nice sequence $\langle \alpha_0, \ldots, \alpha_k \rangle$ can be used to represent μ_α in M_α, in the usual sense that for a function $h \in V$,

$$\mu_\alpha = j_\alpha(h)\left(j_{1,\alpha}\left(\vec{\kappa}\right), j_{\alpha_0,\alpha}\left(\vec{\mu}_{\alpha_0}\right), \ldots, j_{\alpha_k,\alpha}\left(\vec{\mu}_{\alpha_k}\right)\right)$$

Apply Multivariable Fusion. For every $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle$, let–

$$\begin{aligned} e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) = \{ & r \in P \setminus \nu_k \colon r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \parallel X\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \in U^{\times}_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)}, \\ & \text{if it decides positively, then } r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \Vdash \underset{\sim}{A}^r_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \subseteq \\ & X\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right); \text{ else, } r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \Vdash \underset{\sim}{A}^r_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \text{ is disjoint} \\ & \text{from } X\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right). \text{ Moreover, } r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \parallel \text{lh}\left(t^r_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)}\right) > n^*, \\ & \text{and if it decides positively, then there exists a bounded subset} \\ & A\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \subseteq h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \text{ for which } r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \Vdash \text{ the } n^*\text{-th} \\ & \text{element of } t^r_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \text{ belongs to } A\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \} \end{aligned}$$

Applying the same tools above, there exists a condition $p^* \in G$ and a bounded subset $A^*\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$, such that–

$$\begin{aligned} \{ & \xi < \kappa \colon p^* \restriction_{h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)} \Vdash \text{if lh}\left(t^r_{h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)}\right) \geq n^* \text{ then the} \\ & n^*\text{-th element in the Prikry sequence of} \\ & h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right) \text{ belongs to } A^*\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)\} \in W \end{aligned}$$

Since $A^*\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$ is bounded in $h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$, it follows that–

$$\{\xi < \kappa \colon p^* \restriction_{h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)} \Vdash \mathrm{lh}\left(t^r_{h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)}\right) < n^*\} \in W$$

By the choice of p^*, it follows that for a set of ξ-s in W,

$$p^* \restriction_{h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)} \parallel X\left(\xi, \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \in U^{\times}\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$$

we argue that for a set of ξ-s in W, the decision is positive. Indeed, otherwise, it holds in $M[H]$ that–

$$\mu_\alpha = [\xi \mapsto \mu_\alpha(\xi)]_W \in \left[\xi \mapsto h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right) \setminus X\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)\right]_W = k_\alpha(\mu_\alpha \backslash X)$$

contradicting the choice of X. Thus, for a set of ξ-s in W,

$$p^* \restriction_{h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)} \Vdash X\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right) \in U^{\times}\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$$

recall that $U^0 = U^{\times} \cap V$; hence–

$$p^* \restriction_{h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)} \Vdash X\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right) \in U^{0}\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$$

and thus, in $M[H]$, $k_\alpha(X) \in k_\alpha\left(U^0_{\mu_\alpha}\right)$. In particular, in M_α, $X \in U^0_{\mu_\alpha}$.

We now proceed to the proof for $U^j_{\mu_\alpha}$ for every $1 \le j \le m_\alpha - 1$. Assume that $\mu^{*j}_\alpha \in k_\alpha(X)$, and recall that μ^{*j}_α is the j-th element in $d^{-1}(\mu_\alpha)$. We repeat the same argument above. First, define–

$$\begin{aligned} e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) = \{r \in P \setminus \nu_k \colon\, & r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \parallel X\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \in U^j_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)}, \\ & \text{if it decides positively, then } r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \Vdash \underset{\sim}{A}^r_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \subseteq \\ & d'' X\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right);\ \text{else, } r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \Vdash \pi^{-1}_{j,0}{}'' \underset{\sim}{A}^r_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \text{ is disjoint} \\ & \text{from } X\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right). \text{ Moreover, } r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \parallel \mathrm{lh}\left(t^r_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)}\right) > n^*, \\ & \text{and if it decides positively, then there exists a bounded subset} \\ & A\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \subseteq h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \text{ for which } r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \Vdash \text{ the } n^*\text{-th} \\ & \text{element of } t^r_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \text{ belongs to } A\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)\} \end{aligned}$$

Now we argue as before, and claim that–

$$p^* \restriction_{h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)} \Vdash X\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right) \in U^{j}\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$$

indeed, otherwise, there exists a set of ξ-s in W for which–

$$\mu_\alpha^j(\xi) = \pi_{j,0}^{-1}\left(\mu_\alpha(\xi)\right) \notin X\left(\vec{\mu}_0^*(\xi), \vec{\mu}_{\alpha_1}^*(\xi), \ldots, \vec{\mu}_{\alpha_k}^*(\xi)\right)$$

contradicting the fact that $\mu_\alpha^j \in k_\alpha(X)$. It follows that, in $M[H]$, $k_\alpha(X) \in k_\alpha\left(U^j_{\mu_\alpha}\right)$, and so $X \in U^j_{\mu_\alpha}$. □

Corollary 3.4.22. $\mathcal{E}_\alpha = \{X \subseteq [\mu_\alpha]^{m_\alpha} : \langle \mu_\alpha, \mu_\alpha^{*1}, \ldots, \mu_\alpha^{*m_\alpha - 1}\rangle \in k_\alpha(X)\} \cap M_\alpha$.

Proof. It suffices to prove that–

$$\mathcal{E}_\alpha \subseteq \{X \subseteq [\mu_\alpha]^{m_\alpha} : \langle \mu_\alpha, \mu_\alpha^{*1}, \ldots, \mu_\alpha^{*m_\alpha - 1}\rangle \in k_\alpha(X)\} \cap M_\alpha$$

Start from $X \in \mathcal{E}_\alpha$. Then there are sets $X_0 \in U^0_{\mu_\alpha}, \ldots, X_{m_\alpha - 1} \in U^{m_\alpha - 1}_{\mu_\alpha}$ such that the set of increasing sequences in $X_0 \times \ldots \times X_{m-1}$ is contained in X. Thus every increasing sequence in $k_\alpha(X_0) \times \ldots \times k_\alpha(X_{m-1})$ belongs to $k_\alpha(X)$, and by the previous lemma, $\langle \mu_\alpha, \ldots, \mu_\alpha^{*m_\alpha - 1}\rangle \in k_\alpha(X)$, as desired. □

This concludes the proof of properties $(A) - (F)$ from the beginning of the section. We now focus on the proof of theorem 3.1.2.

Recall that $\kappa^* = j_U(\kappa)$. Note that $\kappa^* = j_{\kappa^*}(\kappa)$, since κ is mapped to κ^* after the first step in the iteration, and every step after it is taken on a measurable μ_α^j below κ^*. Moreover, $\sup\{\mu_\alpha \colon \alpha < \kappa^*\} = \kappa^*$ and thus $\mathrm{crit}(k_{\kappa^*}) \geq \kappa^*$. Let us use the above properties and argue that the induction halts after κ^*-many steps.

Proof of Theorem 3.1.2. Since $j_W \restriction_V = j_{\kappa^*} \circ k_{\kappa^*}$, it suffices to prove that $k_{\kappa^*} \colon M_{\kappa^*} \to M$ is the identity function. In particular, it will follow that $j_W \restriction_V = j_{\kappa^*}$, $M = M_{\kappa^*}$ and $j_W(\kappa) = j_{\kappa^*}(\kappa) = j_U(\kappa)$.

We argue that for every ordinal η, $\eta \in \mathrm{Im}(k_{\kappa^*})$. Fix such η and let $g \in V[G]$ be a function such that $[g]_{W^*} = \eta$. Let $\underset{\sim}{g}$ be a $P = P_\kappa$-name for it. For every $\xi < \kappa$, let–

$$e(\xi) = \{r \in P \setminus \xi \colon \text{for some } A \subseteq \kappa \text{ with } |A| < \kappa,\ r \Vdash \underset{\sim}{g}(\xi) \in A\}$$

$e(\xi)$ is $\leq^*$-dense open by lemma 3.2.8. By Fusion, there exists $p \in G$ such that for every $\xi < \kappa$,

$$p \restriction_\xi \Vdash \text{for some } A \subseteq \kappa \text{ with } |A| < \kappa,\ (p \setminus \xi)^{-\xi} \Vdash \underset{\sim}{g}(\xi) \in A$$

Fix $\xi < \kappa$ and let $\underset{\sim}{A}$ be a P_ξ-name for the above subset $A \subseteq \kappa$. Let–

$$A(\xi) = \{\gamma < \kappa \colon \exists q \geq p \restriction_\xi,\ q \Vdash \gamma \in \underset{\sim}{A}\}$$

Then for every $\xi < \kappa$, $|A(\xi)| < \kappa$, and $p^{-\xi} \Vdash \underset{\sim}{g}(\xi) \in A(\xi)$. Recall that for every $p \in G$, $(j_W(p))^{-[Id]_{W^*}} \in H$. Thus, in $M[H]$,

$$[g]_{W^*} \in j_{W^*}(\xi \mapsto A(\xi))([Id]_{W^*}) = k_{\kappa^*}\left(\ j_{\kappa^*}(\xi \mapsto A(\xi))(j_{1,\kappa^*}(\kappa))\ \right)$$

but $|j_{\kappa^*}(\xi \mapsto A(\xi))(j_{1,\kappa^*}(\kappa))| < j_{\kappa^*}(\kappa) = \kappa^* \leq \mathrm{crit}(k_{\kappa^*})$, and thus $[g]_{W^*} \in \mathrm{Im}(k_{\kappa^*})$. □

Lemma 3.4.23. *Fix $\alpha < \kappa^*$ and denote $m = m_\alpha$. Let $0 < j \leq m$. Then μ_α^{*j} is measurable in M, and its Prikry sequence in $M[H]$ is the sequence of critical points obtained by iterating the measure $U^{j-1}_{\mu_\alpha}$ over M_α.*

Proof. First, by lemma 3.4.17, μ_α appears in the Prikry sequence of μ_α^{*j}. Let λ be the element after it in this Prikry sequence, and let us argue that $\lambda = j_{U^{j-1}_{\mu_\alpha}}(\mu_\alpha) = \mu_\alpha^{j-1}$. Since $j_{U^{j-1}_{\mu_\alpha}}(\mu_\alpha)$ is measurable in $M_{\alpha+1}$ and has cofinality above κ in V, there exists $\beta > \alpha$ such that $\mu_\beta = \mu_\alpha^j$; Also, $k_\beta(\mu_\beta) = k_\beta\left(j_{\alpha+1,\beta}\left(\mu_\alpha^j\right)\right) = k_{\alpha+1}\left(\mu_\alpha^j\right) = \mu_\alpha^{*j}$, and thus $\mu_\beta = \mu_\alpha^j$ appears as an element in the Prikry sequence of μ_α^{*j}. Thus, $\lambda \leq \mu_\alpha^j$, and it suffices to prove that $\lambda = \mu_\alpha^j$. Assume for contradiction that $\lambda < \mu_\alpha^{j-1} = j_{U^{j-1}_{\mu_\alpha}}(\mu_\alpha)$.

In M_α write–

$$\mu_\alpha = j_\alpha(h)\left(j_{1,\alpha}\left([Id]_0^{'}\right), j_{\alpha_1+1,\alpha}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right)$$

and–

$$\lambda = j_{\alpha+1}(g)\left(j_{1,\alpha}\left([Id]_0^{'}\right), j_{\alpha_1+1,\alpha}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right), [Id]_\alpha\right)$$

for some functions f, g in V. Recall that $[Id]_\alpha = \langle \mu_\alpha, \ldots, \mu_\alpha^{j-1}, \ldots, \mu_\alpha^{m_\alpha - 1}\rangle$, so we can assume that for every $\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k, \vec{\nu} = \langle \nu_0, \ldots, \nu_{m-1}\rangle$,

$$g\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k, \langle \nu^0, \ldots, \nu^{j-1}, \ldots, \nu^{m-1}\rangle\right) < \min\{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right), \nu^{j-1}\}$$

Let t^* be the initial segment of the Prikry sequence of $k_\alpha(\mu_\alpha)$ which consists of all the ordinals below μ_α. Fix a function $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle \mapsto t^*\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ which represents t^* in M_α (as in lemma 3.4.17).

For simplicity, we adopt the following notation below: whenever $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle$ are fixed, let $h = h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$. Also, for every $\nu < h$, we denote $d^{-1}(\nu) = \langle \nu^1, \ldots, \nu^{m-1} \rangle$ (whenever $m \neq 1$). We also denote $\nu^0 = \nu$ and $\vec{\nu} = \langle \nu^0, \ldots, \nu^{m-1} \rangle$. Let $C = C\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ be the club of closure points of $\nu_0 \mapsto g\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k, \vec{\nu}\right)$ (this is a club in h. We remark that it is necessary in the proof below only in the case where $j = 1$).

We now apply the Multivariable Fusion lemma. Fix $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle$, and let–

$$e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) = \{r \in P \setminus \nu_k \colon \text{for every } \nu \in \underset{\sim}{A}^r_h, \\ A^r_h \setminus \nu \subseteq \left(h \setminus \nu^{j-1}\right) \cap C\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)\}$$

First let us consider the case where $j > 1$. There exists $p \in G$ such that for a set of $\xi < \kappa$ in W, the condition $p^\frown \langle \vec{\xi}, \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi) \rangle$ forces that the element which appears after $\mu_\alpha(\xi)$ in the Prikry sequence of $h\left(\vec{\xi}, \vec{\mu}^*_{\alpha_0}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$ is strictly greater then $\mu^{*j-1}(\xi)$. Thus, in $M[H]$,

$$\lambda > \mu^{*j-1} > j_W(g)\left(\vec{\kappa}, \vec{\mu}^*_{\alpha_0}, \ldots, \vec{\mu}^*_{\alpha_k}, \vec{\mu}^*_\alpha\right) \geq \lambda$$

which is a contradiction.

If $j = 1$, we use the club C defined above: Since $\mu_\alpha < \lambda$, it follows that $\lambda > j_W(g)\left(\vec{\kappa}, \vec{\mu}^*_{\alpha_1}, \ldots, \vec{\mu}^*_{\alpha_k}, \vec{\mu}^*_\alpha\right)$ which is again a contradiction as above. □

Corollary 3.4.24. *In $M[H]$, recall the sequence–*

$$d^{-1}\{\kappa\} = \langle \mu_0^{*1}, \ldots, \mu_0^{*m} \rangle = \langle [Id]_{W^1}, \ldots, [Id]_{W^m} \rangle$$

For every $1 \leq j \leq m$, the cardinal $\mu^{*j} = [Id]_{W^j}$ is measurable in M, and its Prikry sequence in $M[H]$ is the sequence of critical points in the iterated ultrapower, ω-many times, taken with the measure $U^{j-1} = W^{j-1} \cap V \in V$.

Finally, let us prove Theorem 3.1.3 and provide a sufficient condition for definability of $j_W \restriction_V$. Denote–

$$\vec{\mathcal{U}} = \langle U_\xi \colon \xi < \kappa \rangle$$

Recall that for every $\xi \in \Delta$, U_ξ is the measure on ξ in V such that $W_\xi = U_\xi^\times$. We will argue below that if $\vec{\mathcal{U}} \in V$, then $j_W \restriction_V$ is a definable class of V.

Remark 3.4.25. *The sequence $\vec{\mathcal{U}}$ might be external to V, even though every measure in it belongs to V. For instance, we may consider iterations where the measure used at stage $\alpha \in \Delta$ codes generic information about Prikry sequences below α.*

More specifically, Let $\eta = \min \Delta$ be the first measurable, and assume that there are unboundedly many measurables $\zeta < \kappa$ which carry at least η measures of Mitchell order 0 (see section 5 in [8] for a detailed construction of a model satisfying the above assumptions). Let $\langle \zeta_\xi \colon \xi < \kappa \rangle$ be an enumeration of the set of such measurables, and let $\langle U_{\zeta_\xi, \alpha} \colon \alpha < \eta \rangle$ be an enumeration of η-many measures of order 0 on each ζ_ξ.

Denote by $\langle \eta_n \colon n < \omega \rangle$ the Prikry sequence added to η in the iteration. For each $\xi < \kappa$, write $\xi = \xi' + n$ for ξ' limit and $n < \omega$. In the Prikry forcing at stage ζ_ξ in the iteration, use the measure $U^{\times}_{\zeta_\xi, \eta_n} = U^{*}_{\zeta_\xi, \eta_n}$. For every other measurable $\zeta \in \Delta$, use the extension of an arbitrary measure of order 0. Note that the forcing $P = P_\kappa$ generated this way uses only measures of order 0, so the iterated ultrapower is taken with measures $\mathcal{E}_\alpha = U^0_{\mu_\alpha}$, which are a single measure, and not a product of several measures on μ_α, for every $\alpha < \kappa^*$.

Let G be a generic set for the above iteration over V. Clearly $\vec{\mathcal{U}} \notin V$, as it codes the Prikry sequence $\langle \eta_n \colon n < \omega \rangle$ added to η.

Let us argue also that $j_W \restriction_V$ is not a definable class of V. As usual, let $j_W \colon V[G] \to M[H]$ be the ultrapower embedding. Let $\langle \lambda_n \colon n < \omega \rangle$ be an enumeration of the first ω-many measurables of $M_U = M_0$ which carry η-many measures. Then, by the analysis in this section, each λ_n carries a measure $U^{M_U}_{\lambda_n, \eta_n}$, which is iterated ω-many times in order to produce a Prikry sequence for a measurable $\lambda^*_n = j_{1,\kappa^*}(\lambda_n)$ of M. So λ_n remains measurable in M_{λ_n} (as all the steps in the iteration $\langle M_\alpha \colon \alpha < \lambda_n \rangle$ are ultrapowers on measurables below λ_n), and–

$$j_{\lambda_n + 1} = j_{U^{M_{\lambda_n}}_{\lambda_n, \eta_n}} \circ j_{\lambda_n}$$

Here, $U^{M_{\lambda_n}}_{\lambda_n, \eta_n}$ is the η_n-th measure in the enumeration $j_{\lambda_n}(\zeta \mapsto \langle U_{\zeta, \alpha} \colon \alpha < \eta \rangle)(\lambda_n)$ of η-many normal measures of order 0 on λ_n in M_{λ_n}. Note that–

$$j_{1, \lambda_n}\left(U^{M_U}_{\lambda_n, \eta_n}\right) = U^{M_{\lambda_n}}_{\lambda_n, \eta_n}$$

Now, Assume for contradiction that $j_W \restriction_V$ is definable in V. Then the embedding $k \colon M_U \to M$,

$$k([f]_U) = [f]_{W^*} = j_W \restriction_V (f)([Id]_{W^*})$$

is definable in V as well. Fix a set $X \subseteq \lambda_n$ in M_U. Then–

$$\begin{aligned} X \in U^{M_U}_{\lambda_n,\eta_n} &\iff \lambda_n \in j_{U^{M_U}_{\lambda_n,\eta_n}}(X) \\ &\iff j_{1,\lambda_n}(\lambda_n) \in j_{1,\lambda_n}\left(j_{U^{M_U}_{\lambda_n,\eta_n}}(X)\right) \\ &\iff \lambda_n \in j_{1,\lambda_n+1}(X) \\ &\iff \lambda^*_n \in k(X) \end{aligned}$$

and thus $U^{M_U}_{\lambda_n,\eta_n}$, and in particular η_n, can be read from k. Thus, if $j_W \restriction_V$ is definable in V, then so is the sequence $\langle \eta_n : n < \omega \rangle$, which is a contradiction.

Finally. let us remark that it is possible that $j_W \restriction_V$ is definable in V, even though $\vec{\mathcal{U}} \notin V$. As above, let $\langle \eta_n : n < \omega \rangle$ be the Prikry sequence added to the first measurable η. On the first ω-many measurables $\langle \zeta_n : n < \omega \rangle$, choose the measures $\langle W_{\zeta_n} : n < \omega \rangle$ as above, namely $W_{\zeta_n} = U^*_{\zeta_n,\eta_n}$. For every other ζ_ξ ($\omega \le \xi < \kappa$), use the measure derived from the least normal measure of order 0 on ζ_ξ in V, with respect to a prescribed well order $\mathcal{W}$ of V_κ. In this case $\mathcal{U} \notin V$, but $j_W \restriction_V$ is definable: For every $\alpha \le \kappa^*$, the normal measure U_{μ_α} is chosen least, among normal measures of order 0 on μ_α, with respect to the well order $j_\alpha(\mathcal{W})$ (the use of the generic Prikry sequence added to η is done boundedly below κ, and thus does not influence the value of U_{μ_α}).

Proof of Theorem 3.1.3. We begin by proving the following claim:

Claim 3.4.26. *Assume that $\vec{\mathcal{U}} \in V$. Then $\langle \langle U^0_\xi, \dots, U^{m_\xi - 1}_\xi, U^{m_\xi}_\xi = U_\xi \rangle : \xi \in \Delta \rangle \in V$ as well.*

Proof. We prove by induction on $\alpha \in \Delta$ that $\langle \langle U^0_\xi, \dots, U^{m_\xi - 1}_\xi, U^{m_\xi}_\xi \rangle : \xi \in \Delta \cap \alpha \rangle$ is definable over V from U_α and $\langle U_\xi : \xi < \alpha \rangle$.

Fix $\alpha \in \Delta$. If U_α does not concentrate on $\Delta \cap \alpha$, then $m_\alpha = 0$ and $U^{m_\alpha}_\alpha = U_\alpha$. Assume otherwise. Denote $m = j_{U_\alpha}(\langle m_\xi : \xi \in \Delta \cap \alpha \rangle)(\alpha)$. We argue that $m_\alpha = m$. Consider the generic extension $V[G_\alpha]$ up to coordinate α. U_α concentrates on elements $\xi \in \Delta$ for which $m = m_\xi$. Each such $\xi \in \Delta$ satisfies that ξ is the m-th element in $d^{-1}\{d(\xi)\}$. Thus, in $\mathrm{Ult}(V[G_\alpha], U^\times_\alpha)$, $d^{-1}\{\kappa\} = m$ and thus $m(U^\times_\alpha) = m$. Thus indeed $m_\alpha = m$.

By the analysis done in this section (more specifically, corollary 3.4.19 and lemma 3.4.20, applied in $V[G_\alpha]$), for every $0 \le j < m$,

$$U^j_\alpha = \left[\xi \mapsto U^j_\xi\right]_{U_\alpha}$$

(where U_ξ^j exists for a set of ξ-s in U_α, since $j < m$, and the function which maps each $\xi < \alpha$ to U_ξ^j is definable in V by the induction hypothesis). Thus the sequence $\langle U_\alpha^0, \ldots, U_\alpha^{m_\alpha - 1}, U_\alpha^{m_\alpha} = U_\alpha \rangle$ is definable over V from $\langle \langle U_\xi^0, \ldots, U_\xi^{m_\xi - 1}, U_\xi^{m_\xi} \rangle : \xi \in \Delta \cap \alpha \rangle$ and U_α. □

Now let us proceed to the proof of the theorem. We prove by induction on $\alpha \leq \kappa^*$ that $j_\alpha : V \to M_\alpha$ is definable in V. Fix $\alpha < \kappa^*$ and assume that $j_\alpha : V \to M_\alpha$ has been defined in V. Let us define the measure $\mathcal{E}_\alpha$.

We use below the usual notations: for some $\alpha_1 < \ldots < \alpha_k < \alpha$ and $h \in V$,

$$\mu_\alpha = j_\alpha(h) \left(j_{1,\alpha} \left([Id]_0' \right), j_{\alpha_1+1,\alpha} \left([Id]_1 \right), \ldots, j_{\alpha_k+1,\alpha} \left([Id]_k \right) \right)$$

(for sake of definability, we can use the least $\langle \alpha_0, \ldots, \alpha_k \rangle$ and h, taken with respect to a prescribed well order of V_λ for λ high enough). For every $\xi \in \Delta$, let $\mathcal{E}(\xi)$ be the measure on $[\xi]^{m_\xi - 1}$ which corresponds to the sequence–

$$U_\xi^0 \lhd U_\xi^1 \lhd \ldots \lhd U_\xi^{m_\xi - 1}$$

Since $\vec{\mathcal{U}}$ belongs to V, the mapping $\xi \mapsto \mathcal{E}(\xi)$ belongs to V as well, by the claim. By corollary 3.4.19, for every $\alpha < \kappa^*$,

$$\mathcal{E}_\alpha = j_\alpha \left(\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \mapsto \mathcal{E}(h \left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \right)) \right) \left(j_{1,\alpha} \left([Id]_0' \right), j_{\alpha_1+1,\alpha} \left([Id]_1 \right), \ldots, j_{\alpha_k+1,\alpha} \left([Id]_k \right) \right)$$

and this definition can be carried inside V. □

Chapter 4

The Easton Support

4.1 Introduction

In this chapter we consider the Easton support iteration of Prikry forcings. The situation turned out to be radically different from the Nonstationary and Full support iterations. Namely, we show the following:

Theorem 4.1.1. *Let κ be a measurable cardinal with $2^\kappa = \kappa^+$. Let $\langle P_\alpha, \underset{\sim}{Q}_\beta \colon \alpha \leq \kappa, \beta < \kappa \rangle$ be an Easton-support iteration of Prikry type forcing notions.*

Assume that $\Delta \subseteq \kappa$ is unbounded, such that for every $\alpha < \kappa$, $\underset{\sim}{Q}_\alpha$ is forced to be trivial if and only if $\alpha \notin \Delta$. Let $U \in V$ is a normal measure on κ with $\Delta \notin U$, and $i \colon V \to N$ is an elementary embedding, definable in V, such that the following properties hold[1]*:*

1. *$crit(i) = \kappa$.*
2. *${}^\kappa N \subseteq N$.*
3. *$\kappa \notin i(\Delta)$.*
4. *$U = \{X \subseteq \kappa \colon \kappa \in i(X)\}$.*
5. *$|i(\kappa)| = \kappa^+$.*
6. *$\{i(f)(\kappa) \colon f \in V,\ f \colon \kappa \to \kappa\}$ is unbounded in $i(\kappa)$.*

Assume also that every element of N has the form $i(f)(\beta_1, \ldots, \beta_l)$ for some $f \in V$ and $\beta_1 < \ldots < \beta_l < i(\kappa)$.

[1] A typical example of such N is an ultrapower of V by its κ-closed extender, and $i \colon V \to N$ is its embedding.

Then there exists a measure $W \in V[G]$ extending U, such that, denoting $Ult(V[G], W) \simeq M_W[j_W(G)]$, there exists $k \colon N \to M_W$ with $crit(k) > \kappa$ such that $j_W \restriction_V = k \circ i$.

Furthermore, under mild assumptions on the forcings participating in the iteration, there are $(2^\kappa)^+ = \kappa^{++}$ normal measures W as above extending U (see theorem 4.2.2). This generalizes the Kunen-Paris theorem on the number of normal measures [17].

We then focus on the question what can be said about the embedding $k \colon N \to M_W$. In particular, ask whether it is an iteration of N by measures or extenders (without assuming that $V = \mathcal{K}$ is the core model). We will prove in theorem 4.3.2 that this is the case where $P = P_\kappa$ is an iteration of Prikry forcings (under some restrictions on the normal measures used; see subsection 4.3.2). Furthermore, in this case, k is an iterated ultrapower with normal measures only.

4.2 The Forcing

Definition 4.2.1. *An iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \colon \alpha \leq \kappa\ ,\ \beta < \kappa \rangle$ is called an Easton support iteration of Prikry-type forcings if and only if,*

1. *For every $\alpha < \kappa$, the weakest condition in P_α forces that $\langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}_{Q_\alpha}, \underset{\sim}{\leq}^*_{Q_\alpha} \rangle$ is a Prikry type forcing notion.*
2. *For every $\alpha \leq \kappa$ and $p \in P_\alpha$,*
   (a) *p is a function with domain α such that for every $\beta < \alpha$, $p \restriction \beta \in P_\beta$, and $p \restriction \beta \Vdash p(\beta) \in \underset{\sim}{Q}_\beta$.*
   (b) *If $\alpha \leq \kappa$ is inaccessible, then $supp(p) \cap \alpha$ is bounded in α ($supp(p) \subseteq \alpha$ is the set of indices γ on which $p(\gamma)$ is forced to be non-trivial).*

Suppose that $p, q \in P_\alpha$. Then $p \geq q$, which means that p extends q, holds if and only if:

1. *$supp(q) \subseteq supp(p)$.*
2. *For every $\beta \in supp(q)$, $p \restriction \beta \Vdash p(\beta) \geq_\beta q(\beta)$ (where $\geq_\beta$ is the order of Q_β).*
3. *There is a finite subset $b \subseteq supp(q)$, such that for every $\beta \in supp(q) \setminus b$, $p \restriction \beta \Vdash p(\beta) \geq^*_\beta q(\beta)$ (where $\geq^*_\beta$ is the direct extension order of Q_β).*

If $b = \emptyset$, we say that p is a direct extension of q, and denote it by $p \geq^* q$.

The following properties are standard (see [7] for example):

Lemma 4.2.2. *For every $\lambda \leq \kappa$, P_λ satisfies the Prikry property.*

Lemma 4.2.3. *For every $\lambda \leq \kappa$ which is Mahlo, P_λ has the $\lambda - c.c.$.*

Let U be a normal ultrafilter over κ. Let $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ be an Easton support iteration of a Prikry type forcing notions. Suppose that the following hold:

1. There exists an unbounded subset $\Delta \subseteq \kappa$, $\Delta \notin U$, such that, for every $\alpha < \kappa$,
   (a) $\alpha \in \Delta \longrightarrow \Vdash_{P_\alpha} \underset{\sim}{Q}_\alpha$ is nontrivial.
   (b) $\alpha \notin \Delta \longrightarrow \Vdash_{P_\alpha} \underset{\sim}{Q}_\alpha$ is trivial.
2. For every $\alpha < \kappa$, $\Vdash_{P_\alpha} \langle \underset{\sim}{Q}_\alpha, \underset{\sim}{\leq}^*_\alpha \rangle$ is $\alpha-$closed.
3. For every $\alpha \in \Delta$, $\Vdash_{P_\alpha} |\underset{\sim}{Q}_\alpha| < \min(\Delta \setminus \alpha + 1)$.

Let G be a generic subset of $P = P_\kappa$. We would like to analyze the normal measures on κ in $V[G]$ extending U. The standard way to do so appears in [7], we present it here for sake of completeness.

Lemma 4.2.4. *There exists a normal measure $U^* \in V[G]$ on κ which extends U.*

Proof. Let $\langle \underset{\sim}{A}_\alpha \colon \alpha < \kappa^+ \rangle$ be an enumeration, in V, of $P = P_\kappa$-names, such that every $X \in (\mathcal{P}(\kappa))^{V[G]}$ has the form $(\underset{\sim}{A}_\alpha)_G$ for some $\alpha < \kappa^+$. Such list of names exists since $P = P_\kappa$ is $\kappa - c.c.$. Now, construct, in $V[G]$, a $\leq^*$-increasing sequence of conditions $\langle q_\alpha \colon \alpha < \kappa^+ \rangle$, such that, over $N[G]$, $q_\alpha \parallel \kappa \in j_U(\underset{\sim}{A}_\alpha)$. Such a sequence exists since $V[G] \vDash$ "$\langle j_U(P) \setminus \kappa, \leq^* \rangle$ is $\kappa^+ -$ closed."

Let $\langle \underset{\sim}{q}_\alpha \colon \alpha < \kappa^+ \rangle$ be a P-name for the above sequence. Now, define $U^* \supseteq U$ as follows: For every $\alpha < \kappa^+$, $(\underset{\sim}{A}_\alpha)_G \in U^*$ if and only if there exists $p \in G$ and $\alpha < \kappa^+$ such that–

$$p^\frown \underset{\sim}{q}_\alpha \vDash \kappa \in i(\underset{\sim}{A}_\alpha)$$

We argue that U^* defined above is a normal measure which extends U.

Assume that $\delta < \kappa$ and $\langle \underset{\sim}{X}_\alpha \colon \alpha < \delta \rangle$ is a P_κ-name for a partition of κ in $V[G]$. For every $\alpha < \delta$, define–

$$Y_\alpha = \{\beta < \kappa^+ \colon \exists p \in P_\kappa,\ p \Vdash \underset{\sim}{X}_\alpha = \underset{\sim}{A}_\beta\}$$

Since P is $\kappa - c.c.$, $|Y_\alpha| < \kappa$. Denote–

$$Y = \bigcup_{\alpha<\delta} Y_\alpha$$

Then $Y \subseteq \kappa^+$ is a bounded subset. Pick $\alpha^* < \kappa^+$ high enough which bounds Y. Let us argue that there exists $p \in G$ and a unique $\beta < \delta$ such that–

$$p^\frown \underset{\sim}{q}_{\alpha^*} \Vdash \kappa \in j_U\left(\underset{\sim}{A}_\beta\right)$$

and thus $(\underset{\sim}{A}_\beta)_G \in U^*$.

Work in $N[G]$. Note that $\langle A_\beta \colon \beta \in Y\rangle$ covers the sequence $\langle X_\alpha \colon \alpha < \delta\rangle$. Since q_{α^*} is $\leq^*$ above any q_β for $\beta \in Y$,

$$\forall \xi < \alpha, q_{\alpha^*} \parallel \kappa \in i(\underset{\sim}{X}_\xi)$$

Since $\langle i\left(X_\xi\right) : \xi < \delta\rangle$ is a partition of $i(\kappa)$, there exists a unique $\xi^* < \delta$ such that $q_{\alpha^*} \Vdash \kappa \in i\left(\underset{\sim}{A}_{\xi^*}\right)$. Let $p \in G$ be a condition forcing this. Then $p^\frown \underset{\sim}{q}_{\alpha^*} \Vdash \kappa \in i\left(\underset{\sim}{X}_{\xi^*}\right)$, as desired.

A similar argument shows that U^* is normal. Indeed, given a P_κ-name for a regressive function $\underset{\sim}{f} \colon \kappa \to \kappa$, define, for every $\alpha < \kappa$,

$$X_\alpha = \{\xi < \kappa \colon f(\xi) = \alpha\}$$

and proceed as before to find a unique $\alpha < \kappa$ such that $X_\alpha \in U^*$. $\square$

In particular, U can be extended to a normal measure $U^* \in V[G]$, such that the ultrapower embedding $j_{U^*} \colon V[G] \to M[j_{U^*}(G)]$ satisfies that $j_{U^*} \restriction_V = k \circ j_U$, for an embedding $k \colon M_U \to M$ which satisfies $\mathrm{crit}(k) > \kappa$. Indeed, define $k\left([f]_U\right) = [f]_{U^*}$ for every $f \colon \kappa \to V$ in V.

A natural question here is whether this is the only way to generate a normal ultrafilter on κ in $V[G]$. In [8, 9] it was established that this is the case when considering the Nonstationary support iteration. However, this is not true anymore once full support iterations are considered: in [2] and later in [15], iterations of the standard Prikry forcing were considered. It was proved that every normal measure $U \in V$ on κ with $\Delta \notin U$ can be extended to a normal measure $U^* \in V[G]$ similarly as above, but not every normal measure extending U is generated this way; nevertheless, all the normal measures on κ in $V[G]$ were characterized, either as extensions U^* of measures $U \in V$ with $\Delta \notin U$, or as the projections to normal measure of extensions U^* of a normal ultrafilter $U \in V$ with $\Delta \in U$.

It turns out that the picture in the Easton support iteration of Prikry type forcing notions (and even of the standard Prikry forcings) is radically different. Given an elementary embedding $i\colon V \to N$ with critical point κ, definable in V, the normal measure derived from it, $U = \{X \subseteq \kappa\colon \kappa \in i(X)\}$, can be extended to a normal measure $W \in V[G]$ such that $j_W \restriction_V = k \circ i$, for some $k\colon N \to M$ with $\mathrm{crit}(k) > \kappa$. In the case of iterations of the standard Prikry forcing, k is an iterated ultrapower of N by normal measures only (see section 4.3), while $i\colon V \to N$ can be an embedding derived from an extender (as in the formulation of theorem 4.1.1).

Let us demonstrate that, in the Easton support iteration, there are many more possibilities to get normal measures $W \in V[G]$. We show that an arbitrary embedding $i\colon V \to N$ can be used to extend the normal measure U derived from it.

Lemma 4.2.5. *Assume that $i\colon V \to N$ is an elementary embedding definable in V, with crit$(i) = \kappa$, such that $|i(\kappa)| = \kappa^+$, $\kappa \notin i(\Delta)$, $N \subseteq V$ and ${}^{\kappa}N \subseteq N$. Denote–*

$$U = \{X \subseteq \kappa\colon \kappa \in i(X)\}$$

Then G is $i(P)\restriction_\kappa = P$-generic over N, and:

1. *For every $q \in i(P) \setminus \kappa$, there is $H \in V[G]$ with $q \in H$, which is $\langle i(P) \setminus \kappa, \leq^* \rangle$-generic over $N[G]$.*
2. *Given such $H \in V[G]$, define–*

$$U_H = \{(\underset{\sim}{A})_G : \underset{\sim}{A} \text{ is a } P-\text{name for a subset of } \kappa, \text{ and there exists } p \in G * H \text{ such that } p \Vdash \kappa \in i(\underset{\sim}{A})\}$$

   Then U_H is a normal, $\kappa-$complete ultrafilter on κ which extends U.

Proof.

1. We can enumerate, in $V[G]$, all the maximal antichains in $\langle i(P) \setminus \kappa, \leq^*\rangle$ with order type κ^+, by $i(\kappa)$-c.c. of the forcing, and since $V[G] \vDash |i(\kappa)| = \kappa^+$. Note that $\kappa \notin i(\Delta)$, so in the sense of $N[G]$, the forcing $\langle i(P) \setminus \kappa, \leq^*\rangle$ is more than κ-closed. Moreover, since $V \vDash {}^{\kappa}N \subseteq N$, and $P = P_\kappa$ is $\kappa-$c.c., $V[G] \vDash {}^{<\kappa}N[G] \subseteq N[G]$. Therefore, every sequence of length κ of conditions in $i(P) \setminus \kappa$ which belongs to $V[G]$ belongs to $N[G]$ as well. Thus, in the sense of $V[G]$, the forcing $\langle i(P) \setminus \kappa, \leq^*\rangle$ is κ^+-closed.

Starting from any condition in $i(P) \setminus \kappa$, we can construct (in $V[G]$) a sequence of direct extensions of it, meeting every maximal antichain. This sequence generates a $\leq^*$-generic over $N[G]$ for $i(P) \setminus \kappa$, which belongs to $V[G]$.

2. First, we prove that $W = U_H$ is a normal, κ-complete ultrafilter on κ which extends U. It is not hard to verify that W is a filter. We prove that W is a κ-complete ultrafilter. Assume that $\langle X_\alpha \colon \alpha < \delta \rangle$ is a partition of κ, for some $\delta < \kappa$. Work in $N[G]$. Let $D \subseteq i(P) \setminus \kappa$ be the $\leq^*$-dense open set of conditions which decide the unique $\alpha < \delta$ for which $\kappa \in i\left(\underset{\sim}{X}_\alpha\right)$. Then such a statement is forced by some $r \in H$. Let $p \in G$ be a condition which forces that r has this property, and also decides the value of α. Then $p^\frown r \Vdash \kappa \in i\left(X_\alpha\right)$ and thus $X_\alpha \in W$. Normality of W follows by a similar argument, using the dense set of conditions deciding the value of $i(\underset{\sim}{f})(\kappa)$ for a given regressive function $f \colon \kappa \to \kappa$. The argument works since we don't force over κ in N.

□

Remark 4.2.6. *M. Magidor pointed out the following: Assuming that $N \subseteq V$ and $i \colon V \to N$ is definable in $V[G]$, it follows that N is already a class of V. Indeed, pick a formula φ and a parameter $a \in V[G]$ such that for every x, y in V, $\varphi(x, y, a)$ holds in $V[G]$ if and only if $i(x) = y$. For every ordinal α pick a condition $p_\alpha \in G$ which decides the value of the set $\left(V_{i(\alpha)}\right)^N$, which is the set y for which $\varphi\left(V_\alpha, y, a\right)$ holds. Since P is a set forcing, there exists $p^* \in G$ such that, for unboundedly many ordinals α, $p_\alpha = p^*$. Then N can be defined as a class of V using p^*, $N = \bigcup\{y \colon \exists \alpha \in ON,\ p^* \Vdash \varphi\left(V_\alpha, y, \underset{\sim}{a}\right)\}$.*

In general, the settings of lemma 4.2.5 are not enough ensure that $j_{U_H} \restriction_V = k \circ i$ for some k with $\mathrm{crit}(k) > \kappa$. For instance, given a normal measure U on κ in V with $\Delta \notin U$, the embedding $i = j_{U^2}$ satisfies the settings of lemma 4.2.5, but cannot be used to extend U to a measure U_H for which $j_{U_H} = k \circ i$ for some embedding k with $\mathrm{crit}(k) > \kappa$. This follows since i fails to satisfy clause 3 in the next claim:

Proposition 4.2.7. *Assume that $U \in V$ is a normal measure on κ, $W \in V[G]$ is a normal measure which extends U, $i \colon V \to N$ is an elementary embedding and $j_W \restriction_V = k \circ i$ for some $k \colon N \to M$ with crit$(k) > \kappa$. Then–*

1. $\{X \subseteq \kappa \colon \kappa \in i(X)\} = U$.

2. $|i(\kappa)| = \kappa^{+}$.

3. $\{i(f)(\kappa) \colon f \in V,\ f : \kappa \to \kappa\}$ *is unbounded in* $i(\kappa)$.

Proof.

1. $\{X \subseteq \kappa \colon \kappa \in i(X)\} = U$: Indeed, let $X \subseteq \kappa$ in V with $\kappa \in i(X)$. By applying $k \colon N \to M$ it follows that $\kappa \in j_W(X)$ and hence $X \in W$. Since $X \in V$ and $U = W \cap V$, it follows that $X \in U$.

2. $|i(\kappa)| = \kappa^{+}$: This holds since, in $V[G]$, $|j_W(\kappa)| = 2^{\kappa} = \kappa^{+}$ (since, in V, $2^{\kappa} = \kappa^{+}$), and $i(\kappa) \leq j_W(\kappa)$.

3. $\{i(f)(\kappa) | f : \kappa \to \kappa\}$ is unbounded in $i(\kappa)$: Given $\beta < i(\kappa)$, let $f \in V[G]$ be a function such that $[f]_W = k(\beta)$. Since $k(\beta) < k(i(\kappa)) = j_W(\kappa)$, we can assume that $f \colon \kappa \to \kappa$. The Easton support ensures that there exists $g \colon \kappa \to \kappa$ in V which dominates f. Thus $i(g)(\kappa) \geq \beta$ (indeed, by applying $k \colon N \to M$ on both sides, this is equivalent to $j_W(g)(\kappa) \geq k(\beta) = [f]_W$, which holds, since g dominates f. Note that, when applying k, we used the fact that $\text{crit}(k) > \kappa$).

□

Theorem 4.1.1 will be proved by a sequence of lemmata, concluded in lemma 4.2.14. The main idea in the proof of theorem 4.1.1 is to add representing functions for all the generators of i above κ. This is needed since $j_W \restriction_V$ has a single generator κ.

Definition 4.2.1. *An ordinal* β *is called a generator of* $i \colon V \to N$ *if there are no* $n < \omega$*, ordinals* $\beta_1, \ldots, \beta_n$ *below* β *and a function* $f \in V$ *such that* $\beta = i(f)(\beta_1, \ldots, \beta_n)$.

In the next lemma we construct a function $\alpha \mapsto \theta_\alpha$ in $V[G]$, which will be utilized, alongside functions in V, to represent the generators of i in $\text{Ult}(V[G], W)$.

Lemma 4.2.8. *There exists a* P_κ*-name for a sequence of ordinals,* $\langle \theta_\alpha \colon \alpha < \kappa \rangle$*, such that the following property holds:*

1. *For every $\beta < \kappa$ and $p \in P_\kappa$, there is $\alpha_0 < \kappa$ such that for every $\alpha \geq \alpha_0$ there exists $p^* \geq^* p$ such that $p^* \Vdash \ \theta_\alpha = \beta$.*

2. *For every $\alpha < \kappa$ and condition $p \in P_\kappa$, there exists a condition $p^* \geq^* p$ which decides the value of $\underset{\sim}{\theta}_\alpha$.*

Remark 4.2.9. *When iterating Prikry forcings, the natural candidate for the function $\alpha \mapsto \theta_\alpha$ is the function which maps every $\alpha \in \Delta$ to the first element in its Prikry sequence (this function does not have domain κ, but this can be fixed by defining the function of elements outside of Δ as the value of the first element of Δ above them). The main problem with such a function is that it fails to satisfy clause 2 of the lemma (from density, every $\leq^*$-generic set has some $\alpha \in \Delta$ for which it does not decide the first element in its Prikry sequence). However, we can base the function $\alpha \mapsto \theta_\alpha$ on $\alpha \mapsto$ the first element of the Prikry sequence of α. There will be still many places of disagreement between them, but for every given $\beta < \kappa$ there will be $\alpha < \kappa$ such that β is the first element of the Prikry sequence of α.*

Proof. For every $\alpha < \kappa$, let $\tau_\alpha < \kappa$ be the least ordinal such that $P \restriction_{(\alpha,\tau_\alpha)}$ is not $\alpha - c.c.$. We will argue below that such $\tau_\alpha < \kappa$ exists, but first, let us show that this suffices: Pick an unbounded subset $X \subseteq \kappa$, such that, for every $\alpha, \alpha' \in X$,

$$\alpha < \alpha' \implies \tau_\alpha < \tau_{\alpha'}$$

(for instance, let X be the club of closure points of the function $\alpha \mapsto \tau_\alpha$). Enumerate $X = \langle x_\alpha \colon \alpha \in \Delta\rangle$. For every $\alpha \in \Delta$, let $\langle q_{x_\alpha,\xi} \colon \xi < x_\alpha\rangle$ be an antichain in $P_{(x_\alpha,\tau_{x_\alpha})}$ of cardinality α. Define θ_α to be the unique ordinal $\xi < x_\alpha$ for which $q_{x_\alpha,\xi} \in G \restriction_{(x_\alpha,\tau_{x_\alpha})}$ (if there is no such ξ, which is possible since the antichain is not necessarily maximal, set $\theta_\alpha = 0$).

Now, given $\beta < \kappa$ and a condition $p \in P_\kappa$, pick first $\alpha \in \Delta$ for which x_α bounds the support of p. Direct extend p to p^* such that $p^* \restriction_{(x_\alpha,\tau_{x_\alpha})} = q_{x_\alpha,\beta}$. Then[2] by our definition, p^* forces that $\theta_\alpha = \beta$.

Let us prove now that for every $\alpha < \kappa$ and condition $p \in P_\kappa$, there exists $p^* \geq^* p$ which decides the value of θ_α.

We will direct extend p is the interval $(x_\alpha, \tau_{x_\alpha})$, x_α-many times, to decide whether $q_{x_\alpha,\xi} \in G \restriction_{(x_\alpha,\tau_{x_\alpha})}$, for every $\xi < x_\alpha$. Note that this is possible since $\langle P \restriction_{(x_\alpha,\tau_{x_\alpha})}, \leq^* \rangle$ is more than x_α-

[2] It is crucial here that the Easton support is used.

closed. Let $p^* \geq^* p$ be the obtained condition. Then either there exists $\xi < x_\alpha$ such that p^* forces that $q_{x_\alpha,\xi}$ is in the generic, and then $p^* \Vdash \underset{\sim}{\theta}_\alpha = \xi$; or, there is no such ξ, and then $p^* \Vdash \underset{\sim}{\theta}_\alpha = 0$.

Let us argue now that indeed, for every $\alpha < \kappa$ there exists $\tau_\alpha < \kappa$ such that $P \restriction_{(\alpha,\tau_\alpha)}$ is not $\alpha - c.c.$: Pick τ_α such that there are α-many elements of Δ in the interval (α, τ_α). Let $\langle \tau_{\alpha,\xi} \colon \xi < \alpha \rangle$ be an enumeration of the first α-many elements in $(\alpha, \tau_\alpha) \cap \Delta$. For every $\xi < \alpha$, let $\underset{\sim}{x}_\xi, \underset{\sim}{y}_\xi$ be $P_{\tau_{\alpha,\xi}}$-names, forced by $0_{P_{\tau_{\alpha,\xi}}}$ to be pair of incompatible elements of $\underset{\sim}{Q}_{\tau_{\alpha,\xi}}$. Such a pair exists since $\underset{\sim}{Q}_{\tau_{\alpha,\xi}}$ is nontrivial.

Now, for every $\sigma \in 2^\alpha$, let $p_\sigma \in P \restriction_{(\alpha,\tau_\alpha)}$ be the condition which satisfies, for every $\xi < \alpha$, that–

$$p_\sigma \restriction_\xi \Vdash p_\sigma(\xi) = \begin{cases} \underset{\sim}{x}_\xi & \text{If } \sigma(\xi) = 0 \\ \underset{\sim}{y}_\xi & \text{If } \sigma(\xi) = 1 \end{cases}$$

Note that τ_α is the limit of the first α many elements above α in Δ, and thus τ_α is singular, so the support of a condition in $P = P_\kappa$ may be unbounded in τ_α.

Then $\langle p_\sigma \colon \sigma \in 2^{<\alpha} \rangle$ is an antichain in $P \restriction_{(\alpha,\tau_\alpha)}$ of cardinality at least α. $\square$

Remark 4.2.10. *Given a function $\alpha \mapsto \underset{\sim}{\theta}_\alpha$ as in lemma 4.2.8, we slightly abuse the notation and denote $i(\alpha \mapsto \underset{\sim}{\theta}_\alpha)$ by $\langle \underset{\sim}{\theta}_\alpha \colon \alpha < i(\kappa) \rangle$.*

Lemma 4.2.11. *Under the assumptions of theorem 4.1.1, there exists $H \in V[G]$ which is $\langle i(P) \setminus \kappa, \leq^* \rangle$-generic over $N[G]$, with the following property:*

$$(*) \quad \text{For every generator } \beta \in i(\kappa) \setminus (\kappa+1) \text{ of } i, \text{ there exists a function } f = f_\beta \in V,$$
$$f \colon \kappa \to \kappa \text{ and a condition } q \in H \text{ such that } q \Vdash \beta = i\left(\alpha \mapsto \underset{\sim}{\theta}_{f(\alpha)}\right)(\kappa).$$

where $\langle \underset{\sim}{\theta}_\alpha \colon \alpha < i(\kappa) \rangle$ is as in remark 4.2.10.

Proof. In $V[G]$, let $\langle A_\xi \mid \xi < \kappa^+ \rangle$ be an enumeration of maximal antichains in $i(P)$. Let $\langle \beta_\xi \mid \xi < \kappa^+ \rangle$ be an enumeration of all the generators of i below $i(\kappa)$. Define in $V[G]$ a $\leq^*$ −increasing sequence $\langle r_\xi \mid \xi < \kappa^+ \rangle$. Assume that $\langle r_\xi \colon \xi < \xi^* \rangle$ has been constructed for some $\xi^* < \kappa^+$. Pick a condition r which $\leq^*$ extends all the conditions $\langle r_\xi \colon \xi < \xi^* \rangle$ constructed so far, and, by extending it, assume that r extends a condition in A_{ξ^*}. Finally, let $\alpha_0 < i(\kappa)$ be such that for every $\alpha \geq \alpha_0$ there exists $r^* \geq^* r$ which forces that $i(\xi \mapsto \theta_\xi)(\alpha) = \beta_{\xi^*}$. Pick any $\alpha \geq \alpha_0$ below $i(\kappa)$ which has the form $i(f)(\kappa)$ for some $f = f_{\beta_{\xi^*}} \in V$, and let $r_{\xi^*} \geq^* r$ be a condition which forces that $i(\xi \mapsto \theta_\xi)(\alpha) = \beta_{\xi^*}$.

Finally, let H be the $\leq^*$-generic generated from $\langle r_\xi \colon \xi < \xi^* \rangle$. $\square$

Remark 4.2.12. *Repeating the above argument, we can construct 2^{κ^+}-many distinct generic sets H satisfying property $(*)$, by constructing a binary tree $\langle r_\sigma \colon \sigma \in 2^{<\kappa^+} \rangle$ of conditions, which are $\leq^*$-increasing in each branch, and for each $\sigma \in 2^{<\kappa^+}$, $r_{\sigma^\frown\langle 0\rangle}$ and $r_{\sigma^\frown\langle\rangle}$ are $\leq^*$-incompatible. Assuming $2^{\kappa^+} = \kappa^{++}$, this provides the maximal number of generic sets H in $V[G]$ for $\langle i(P) \setminus \kappa, \leq^* \rangle$ over $N[G]$.*

Below we will define for every such H a measure $U_H \in V[G]$ on κ which extends U; under mild assumptions on the forcing notions $\underset{\sim}{Q}_\alpha$, we will prove that for $H \neq H'$ satisfying property $(*)$, $U_H \neq U_{H'}$ (see theorem 4.2.2). Assuming GCH, this produces the maximal number κ^{++} of normal measures on κ, generalizing the well known result of Kunen and Paris [17].

Remark 4.2.13. *Not every generic set $H \in V[G]$ for $\langle i(P) \setminus \kappa, \leq^* \rangle$ satisfies property $(*)$.*

Indeed, assume that Δ consists only of inaccessibles and $i \colon V \to N$ has a nonempty set of generators in $(\kappa, i(\kappa))$ which is bounded by some ordinal $\eta = i(f)(\kappa)$ below $\min(i(\Delta) \setminus \kappa)$, for some $f \in V$. This holds in the typical case where Δ consists of measurables below κ and i is a (κ, κ^+)-extender (the length of i is κ^+ since i has to satisfy the requirement $|i(\kappa)| = \kappa^+$ of theorem 4.1.1). Let $\sigma \colon M_U \to N$ be the embedding which maps each element $[g]_U$ of M_U to $i(g)(\kappa)$ (here $g \in V$ is any function with domain κ). σ has critical point strictly above κ^+, since $(\kappa^+)^N = \kappa^+$.

In $V[G]$, let $H_U \subseteq j_U(P) \setminus \kappa$ be $\leq^*$-generic over $M_U[G]$. Let $H \subseteq i(P) \setminus \kappa$ be the generic set generated from $\sigma'' H_U$. We argue that H is indeed $\leq^*$-generic over N. Let $D \in N[G]$ be a $\leq^*$-dense open subset of $i(P) \setminus \kappa$. Write $\underset{\sim}{D} = i(F)(\kappa, \beta_1, \ldots, \beta_l)$ for some function $F \in V$, $l < \omega$ and generators $\beta_1, \ldots, \beta_l < i(f)(\kappa)$ of i. We can assume that for every $\xi, \eta_1, \ldots, \eta_l < f(\xi)$, $F(\xi, \eta_1, \ldots, \eta_l) \subseteq P \setminus \xi$ is forced to be $\leq^*$-dense open subset of $P \setminus \xi$. Define, in M_U,

$$D_U = \bigcap_{\gamma_1, \ldots, \gamma_l < j_U(f)(\kappa)} j_U(F)(\kappa, \gamma_1, \ldots, \gamma_l)$$

and note that, since the amount of sequences $\gamma_1, \ldots, \gamma_l < j_U(f)(\kappa)$ in M_U is below $\min(\Delta \setminus \kappa)$, and $\langle j_U(P) \setminus \kappa, \leq^* \rangle$ is more than $\min(j_U(\Delta) \setminus \kappa)$-closed, D_U is $\leq^*$-dense open subset of $j_U(P) \setminus \kappa$. Pick any $q \in H_U \cap D_U$. Then $\sigma(q) \in D \cap H$, since $\sigma(D_U) \subseteq D$.

Since $\sigma'' G * H_U \subseteq G * H$, the embedding $\sigma \colon M_U \to N$ can be lifted to an embedding $\sigma^* \colon M_U[G * H_U] \to N[G * H]$.

Pick now any generator β of i in the interval $(\kappa, i(\kappa))$. We argue that there is no $f \in V$ such that $H \Vdash \beta = i\left(\alpha \mapsto \underset{\sim}{\theta}_{f(\alpha)}\right)(\kappa)$. Indeed, otherwise, by elementarity of σ^*, there exists $\beta^* < j_U(\kappa)$ such that–

$$H_U \Vdash \beta^* = j_U\left(\alpha \mapsto \underset{\sim}{\theta}_{f(\alpha)}\right)(\kappa)$$

Let $g \in V$ be a function such that $\beta^* = j_U(g)(\kappa)$. Then–

$$\beta = \sigma^*(\beta^*) = i(g)(\kappa)$$

contradicting the fact that β is a generator of i.

Given i, N, U as in theorem 4.1.1 and a generic set $H \in V[G]$ for $\langle i(P) \setminus \kappa, \leq^* \rangle$ over $N[G]$, define–

$$U_H = \{(\underset{\sim}{A})_G : \underset{\sim}{A} \text{ is a } P-\text{name for a subset of } \kappa, \text{ and there exists}$$
$$p \in G * H \text{ such that } p \Vdash \kappa \in i(\underset{\sim}{A})\}$$

Then U_H is a normal, κ-complete ultrafilter which extends U. This follows by repeating the argument of lemma 4.2.5.

The model $M_{U_H} \simeq \text{Ult}(V[G], U_H)$ is of the form $M[G^*]$, where M is the image of V and $G^* = j_{U_H}(G)$ is $j_{U_H}(P)-$generic over M in sense of M_{U_H}. We conclude the proof of theorem 4.1.1 by defining an elementary embedding $k\colon N \to M$ and proving that $\text{crit}(k) > \kappa$.

In the next lemma we continue the abuse of notation as in remark 4.2.10, and denote–

$$j_{U_H}(\langle \theta_\xi : \xi \in \Delta \rangle) = \langle \theta_\xi : \xi \in j_{U_H}(\Delta) \rangle$$

Lemma 4.2.14. *Assume the settings of theorem 4.1.1. Suppose that H is a generic set for $\langle i(P) \setminus \kappa, \leq^* \rangle$ over $N[G]$ with the property $(*)$. Define then $k : N \to M$ as follows:*

$$k\left(i(f)(\kappa, \beta_1, ..., \beta_l)\right) = j_{U_H}(f)\left(\kappa, \theta_{[f_{\beta_1}]_{U_H}}, ..., \theta_{[f_{\beta_l}]_{U_H}}\right)$$

For every $l < \omega$, $\beta_1, \ldots, \beta_l < i(\kappa)$ generators of i and $f \in V$ (the functions f_{β_i}, $1 \leq i \leq l$, are as in lemma 4.2.11).

Then $k\colon N \to M$ is elementary, crit$(k) > \kappa$ and $j_{U_H} \restriction_V = k \circ i$.

Proof. Denote $W = U_H$. Let us prove that the embedding k defined above is elementary. Assume that $x, y \in N$. There are functions f, g in V, generators $\beta_1 < \ldots < \beta_l < i(\kappa)$ such that–

$$x = i(f)(\kappa, \beta_1, \ldots, \beta_l), \ y = i(g)(\kappa, \beta_1, \ldots, \beta_l)$$

Assume now that $k(x) = k(y)$, namely–

$$j_W(f)\left(\kappa, \theta_{j_W(f_{\beta_1})(\kappa)}, \dots, \theta_{j_W(f_{\beta_l})(\kappa)}\right) \in j_W(g)\left(\kappa, \theta_{j_W(f_{\beta_1})(\kappa)}, \dots, \theta_{j_W(f_{\beta_l})(\kappa)}\right)$$

Then–

$$\{\xi < \kappa \colon f\left(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}\right) \in g\left(\xi, \theta_{f_{\beta_1}(\xi)}, \dots, \theta_{f_{\beta_l}(\xi)}\right)\} \in W$$

and by the definition of W, there exists $p \in G$ and $r \in H$ such that–

$$p^\frown r \Vdash \kappa \in i\Big(\{\xi < \kappa \colon f\left(\xi, \underset{\sim}{\theta}_{f_{\beta_1}(\xi)}, \dots, \underset{\sim}{\theta}_{f_{\beta_l}(\xi)}\right) \in g\left(\xi, \underset{\sim}{\theta}_{f_{\beta_1}(\xi)}, \dots, \underset{\sim}{\theta}_{f_{\beta_l}(\xi)}\right)\}\Big)$$

By extending $r \in H$ finitely many times, $p^\frown r \Vdash \underset{\sim}{\theta}_{(i(f_{\beta_m})(\kappa))} = \beta_m$ holds for every $1 \leq m \leq k$. Thus, the last equation can be replaced with–

$$p^\frown r \Vdash i(f)\left(\kappa, \beta_1, \dots, \beta_l\right) \in i(g)\left(\kappa, \beta_1, \dots, \beta_l\right)$$

but the forced statement above is entirely in N, and since a condition forces it, it is true in N. Thus–

$$i(f)\left(\kappa, \beta_1, \dots, \beta_l\right) \in i(g)\left(\kappa, \beta_1, \dots, \beta_l\right)$$

as desired. The implication in the other direction is proved similarly.

Clearly $\mathrm{crit}(k) > \kappa$. We finish the proof by showing that $j_W \restriction_V = k \circ i$. Let $x \in V$ and let $c_x \colon \kappa \to V$ be the constant function with value x. Then–

$$k\left(i(x)\right) = k\left(i\left(c_x\right)(\kappa)\right) = j_W\left(c_x\right)(\kappa) = j_W(x)$$

as desired. □

Let us now study the properties of the embedding $k \colon N \to M$. We assume the settings of theorem 4.1.1.

Lemma 4.2.15. *If $\leq = \leq^*$, or at least $\leq_\alpha = \leq^*_\alpha$, for a final segment of $\alpha < \kappa$, then k is the identity and $M = N$.*

Proof. Fix an ordinal η, and let $f \in V[G]$ be a function such that $\eta = [f]_W$. We will prove that $\eta \in \mathrm{Im}(k)$. Indeed, consider the set–

$$\{p \in i(P)/G \mid \exists \tau (p \Vdash i(\underset{\sim}{f})(\kappa) = \tau)\}$$

It is $\leq -$dense in $N[G]$. So, if $\leq = \leq^*$, then H meets it. Thus, there exists a condition $q \in H$, a function $g \in V$ and generators $\beta_1, \ldots, \beta_l$ of i, such that $q \Vdash i\left(\underset{\sim}{f}\right)(\kappa) = i(g)\left(\kappa, \beta_1, \ldots, \beta_l\right)$. Thus, by the definition of W,

$$\{\xi < \kappa \colon f(\xi) = g\left(\xi, \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}\right)\} \in W$$

and thus $\eta = [f]_W = k\left(i(g)\left(\kappa, \beta_1, \ldots, \beta_l\right)\right)$.

□

In general, M should not be equal to N. Thus, for example, they will differ if the Prikry forcing was used unboundedly often below κ.

However, we do not know whether the assumption of 2.15 is necessary.

Question 4.2.16. *Suppose that for unboundedly many $\alpha < \kappa$, $\leq_\alpha \neq \leq^*_\alpha$. Is then $M \neq N$?*

We do not know what are the requirements on the forcings $\underset{\sim}{Q}_\alpha$ for $\alpha \in \Delta$ which imply $M = N$. We conjecture that the requirement should be that there is $\delta < \kappa$, such that every set of ordinals x of $V[G]$ can be covered by a set $y \in V$ of cardinality $\leq |x| + \delta$.

Lemma 4.2.17. *$k''H \subseteq G^* \setminus \kappa$.*

Proof. Let q be in H, and let $p \in G$ be a condition such that $p \Vdash \underset{\sim}{q} \in \underset{\sim}{H}$ (recall that $H \in V[G]$). Clearly,

$$p^\frown \underset{\sim}{q} \Vdash \underset{\sim}{q} \in \Gamma \setminus \kappa$$

where Γ is the canonical $i(P)$-name for the generic set for $i(P)$ over V.

Pick $f : [\kappa]^n \to \kappa, f \in V$ and $\beta_1 < ... < \beta_n < i(\kappa)$ such that $\underset{\sim}{q} = i(f)(\beta_1, ..., \beta_n)$. For every $m, 1 \leq m \leq n$, there are $f_m : \kappa \to \kappa, f_m \in V$ such that $q_{i(f_m)(\kappa), \beta_m} \in H$, namely, $\beta_m = \theta_{i(f_m)(\kappa)}$.

Let us argue that the set–

$$A_q = \{\nu < \kappa \mid f(\theta_{f_1(\nu)}, ..., \theta_{f_n(\nu)}) \in \underset{\sim}{G} \setminus \nu\}$$

is in W. Pick any $q \leq^* q^* \in H$ which $\leq^*$ which forces that $\beta_m = \theta_{i(f_{\beta_m})(\kappa)}$, for every $1 \leq m \leq n$. Recall that–

$$q = i(f)(\beta_1, ..., \beta_n) = i(f)(\theta_{i(f_1)(\kappa)}, ..., \theta_{i(f_n)(\kappa)})$$

and thus $p^\frown \underset{\sim}{q}^* \Vdash \kappa \in i(\underset{\sim}{A}_q)$.

□

The next lemma generalizes a Kunen-Paris result (see remark 4.2.12).

Theorem 4.2.2. *Let $H, H' \in V[G]$ be generic sets for $\langle i(P) \setminus \kappa, \leq^* \rangle$ over $N[G]$. Suppose that H and H' satisfy $(*)$. Assume that for every $\beta < \kappa$, if $q, q' \in Q_\beta$ are incompatible according to the order $\leq^*$, then–*

$$D_\beta(q) = \{r \in Q_\beta \mid r \text{ is } \leq -\text{incompatible with } q\}$$

is $\leq^*$ $-$dense above q', or

$$D_\beta(q') = \{r \in Q_\beta \mid r \text{ is } \leq -\text{incompatible with } q'\}$$

is $\leq^*$ $-$dense above q.[3]

Suppose that $H \neq H'$, then $U_H \neq U_{H'}$.

Remark 4.2.18. *Note that if the Q_β-s are taken to be Prikry forcings, then the above property holds. Indeed, assume that $q = \langle t, A \rangle$ and $q' = \langle t', A' \rangle$ are $\leq^*$-incompatible. Then $t \neq t'$. Assume without loss of generality that t is an end extension of t'. Then $D(q) = \{r \colon r, q \text{ are } \leq^* \text{-incompatible}\}$ is $\leq^*$-dense open above q'. Indeed, pick a condition $\langle t', B \rangle \geq^* \langle t', A \rangle$. Shrink B to the set $B^* = B \setminus (\max(t) + 1)$. Then $\langle t', B^* \rangle \geq^* \langle t', B \rangle$ and is incompatible with $q = \langle t, A \rangle$.*

Proof. Suppose otherwise, i.e. $H \neq H'$, but $U_H = U_{H'} := W$.
Let $k : N \to M$ be the elementary embedding defined from H and $k' : N \to M$ from H'.

Claim 4.2.19. *$k \neq k'$.*

Proof. Assume for contradiction that $k = k'$. Thus, by Lemma 4.2.17, every pair of elements from H, H' are $\leq$-compatible. We will argue that this implies that $H = H'$. It suffices to prove that every pair of conditions $q \in H, q' \in H'$ are $\leq^*$-compatible.

Assume otherwise. Let $\alpha < \kappa$ be the least ordinal such that there are pair of conditions $q \in H, q' \in H'$ for which $q \restriction_\alpha, q' \restriction_\alpha$ are $\leq^*$-incompatible. α cannot be limit, since $\leq^*$-compatibility of all the initial segments of q, q' below α implies that $q \restriction_\alpha$ and $q' \restriction_\alpha$ are $\leq^*$-compatible themselves (if α is inaccessible, this is clear since the support of q, q' is bounded in α; if the supports of q, q'

[3] This type of condition usually holds. For example, if we iterate Prikry forcings, then just shrinking sets of measure one will produce such type of incomparability.

are unbounded in α, just intersect sets of measure one to find a common direct extension). Thus $\alpha = \beta + 1$ is successor, and $q(\beta)$, $q'(\beta)$ are $\leq^*$-incompatible. By the property of the forcing Q_β, without loss of generality, $D_\beta(q)$ is $\leq^*$-dense open above q'. Since $q' \in H'(\beta)$, q' can be extended to a condition $r \in H'$, such that $r(\beta) \in D_\beta(q)$. In particular, $q \in H$, $r \in H'$ are $\leq$-incompatible, which is a contradiction. $\square$ of claim 4.2.19.

Since $k \neq k'$, there exists a generator β of i such that $k(\beta) \neq k'(\beta)$. Pick the least such generator β.

Claim 4.2.20. *For every generator $\beta' < \beta$ of i, there exists a function $f_{\beta'} \in V$ such that each generic H, H' has a condition which forces that $\beta' = \underset{\sim}{\theta}_{i(f_{\beta'})(\kappa)}$.*

Proof. Let f, f' be functions such that some condition in H forces that $\beta = \underset{\sim}{\theta}_{i(f)(\kappa)}$, and some condition in H' forces that $\beta' = \underset{\sim}{\theta}_{i(f')(\kappa)}$. Let $q \in H$ be a condition which decides the statement $\underset{\sim}{\theta}_{i(f)(\kappa)} = \underset{\sim}{\theta}_{i(f')(\kappa)}$ and assume for contradiction that it is decided negatively. By applying k, $k(q) \in j_W(G)$ forces that–

$$\underset{\sim}{\theta}_{[f]_W} \neq \underset{\sim}{\theta}_{[f']_W}$$

namely $k(\beta') \neq k'(\beta')$, contradicting the minimality of β. $\square$ of claim 4.2.20.

Recall now that $k(\beta) \neq k'(\beta)$. Thus, there are two distinct functions f, f' in V such that–

1. Some condition in H forces that $\beta = \underset{\sim}{\theta}_{i(f)(\kappa)}$.
2. Some condition in H' forces that $\beta = \underset{\sim}{\theta}_{i(f')(\kappa)}$.
3. Without loss of generality, $\{\xi < \kappa \colon \theta_{f'(\xi)} < \theta_{f(\xi)}\} \in W$.

By property (2) of the names $\langle \underset{\sim}{\theta}_\alpha \colon \alpha < \kappa \rangle$, presented in lemma 4.2.8, there exists an ordinal β' such that some condition in H forces that $\underset{\sim}{\theta}_{i(f')(\kappa)} = \beta'$. By the above assumptions, $\beta' < \beta$.

We argue that β' is a generator of i as well. This will finish the proof: once we prove that β' is a generator of i, it follows from claim 4.2.20 that $\theta_{i(f')(\kappa)}$ represents β' in the sense of both generics, H, H'. However, in the sense of H', it represents β, which is a contradiction.

Assume for contradiction that β' is not a generator of i. Then there is a function $g \in V$ and $\beta_1, \ldots, \beta_l$ below β', such that $\beta' = i(h)(\kappa, \beta_1, \ldots, \beta_l)$. Since H forces that $\beta' = \underset{\sim}{\theta}_{i(f')(\kappa)}$, it follows that–

$$\{\xi < \kappa \colon g\left(\xi, \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}\right) = \theta_{f'(\xi)}\} \in U_H = W$$

Thus the same set belongs to $U_{H'}$. Therefore, H' forces that–

$$\beta = \theta_{i(f')(\kappa)} = i(g)\left(\kappa, \beta_1, \ldots, \beta_l\right)$$

contradicting the fact that β is a generator of i (note that we used claim 4.2.20 when arguing that the generators β_i, $1 \leq i \leq l$, are represented the same way in the sense of H, H'). □

Definition 4.2.3. *A measure $W \in V[G]$ is called simply generated if $W = U_H$ for some $U \in V$, where H is generic for $\langle j_U(P) \setminus \kappa, \leq^* \rangle$ over $M_U[G]$.*

Remark 4.2.21. *Given a simply generated normal measure $W \in V[G]$ as above, with $\Delta \notin W$, the parameters U and H are uniquely defined from it*[4]*. Indeed, we will prove in the next lemma that $U = W \cap V$ belongs to V, and is a normal measure there with $\Delta \notin U$, Now, assume that there are H, H', generic over $M_U[G]$ for $\langle j_U(P) \setminus \kappa, \leq^* \rangle$, with $W = U_H = U_{H'}$. Then H, H' satisfy the conditions of lemma 4.2.2 (since j_U has no generators other than κ). Thus, by the theorem, $H = H'$.*

Given $W \in V[G]$ normal on κ (which is not necessarily simply generated), we can say the following:

Lemma 4.2.22. *Every normal measure $W \in V[G]$ on κ extends a measure $U = W \cap V \in V$.*

Proof. First, let us argue that $U = W \cap V$ belongs to V. By [8], it suffices to prove that there are no new fresh unbounded subsets of cardinals in the interval $\left[\kappa, (2^\kappa)^V\right] = [\kappa, \kappa^+]$. Thus, it suffices to prove the following pair of claims:

Claim 4.2.23. *$P = P_\kappa$ does not add fresh unbounded subsets to κ.*

Proof. The fact that there are no fresh unbounded subsets of κ follows essentially from the facts that $2^\kappa = \kappa^+$, and that there exists a normal measure on κ in $V[G]$: Given a normal measure $U \in V$ with $\Delta \notin U$, take any $U^* \in V[G]$ which extends it. Given a fresh unbounded $A \subseteq \kappa$, $A = j_{U^*}(A) \cap \kappa$ and thus, by elementarity, A belongs to the ground model M of $\mathrm{Ult}(V[G], U^*)$. Now set $k_U \colon M_U \to M$ to be the function which maps $[f]_U$ to $[f]_{U^*}$. Then k_U is a well defined elementary embedding since $U \subseteq U^*$, and $\mathrm{crit}(k_U) > \kappa$ by normality of U^*. Since $2^\kappa = \kappa^+$ holds

[4] Note that when iterating Prikry forcings, $\Delta \notin W$ holds for every normal measure $W \in V[G]$ on κ, since every such W concentrates on regulars.

in M, k_U maps the sequence of subsets of κ to itself, and thus every subset of κ which belongs to M, already belongs to V. So the above set A belongs to V, which is a contradiction. $\square$

Claim 4.2.24. *For every measurable (in V) $\lambda \leq \kappa$, P_λ doesn't add fresh unbounded subsets of λ^+. In particular, P_κ does not add fresh subsets to λ^+.*

Proof. Let $f \in V[G]$ be the characteristic function of a fresh unbounded subset of λ^+. Let $\underset{\sim}{f}$ be a P_λ-name and assume that $p \in P$ forces that $\underset{\sim}{f}$ is fresh.

Let $G \subseteq P_\lambda$ be generic over V. For every $\xi < \lambda^+$, let $p_\xi \in G$ be a condition which decides $\underset{\sim}{f} \restriction_\xi$. For every $\xi < \lambda^+$ there exists $\alpha_\xi < \lambda$ such that the support of p_ξ is bounded by α_ξ. Let $A \subseteq \lambda^+$ and $\alpha^* < \lambda$ be such that $|A| = \lambda^+$ and $\alpha_\xi = \alpha^*$ for every $\xi \in A$.

By shrinking $A \subseteq \lambda^+$ even further (to a set of cardinality λ^+), we can assume that there exists $q^* \in P_\lambda$ such that, for every $\xi \in A$, $p_\xi \restriction_{\alpha^*} = q^* \restriction_{\alpha^*}$, and $q^* \restriction_{[\alpha^*, \lambda)}$ is trivial.

Let $h = \bigcup\{g \colon \exists \xi < \lambda \ q^* \Vdash \underset{\sim}{f} \restriction_\xi = g\}$. Clearly, $h \colon \lambda^+ \to 2$ is a function and $q^* \Vdash \underset{\sim}{f} = \check{h}$. $\square$

$\square$ of lemma 4.2.22.

4.3 The Structure of $j_W \restriction_V$

4.3.1 Properties of k

We continue and use the notations of theorem 4.1.1. We first state the following lemma.

Lemma 4.3.1. *Let $P = P_\kappa$ be an Easton support iteration of Prikry type forcings, and $i \colon V \to N$, $\Delta \subseteq \kappa$, $U \in V$, $W \in V[G]$ and $k \colon N \to M$ be as in section 4.2.*

Assume that there are no elements in $(\kappa, crit(k)) \cap i(\Delta)$. Then $crit(k) \in i(\Delta)$, namely, it is the least element above κ in $i(\Delta)$.

Remark 4.3.2. *The assumption $(\kappa, crit(k)) \cap i(\Delta) = \emptyset$ holds in the typical case where $P = P_\kappa$ is an iteration of Prikry forcings. Indeed, assume, by contradiction, that there exists $\mu \in (\kappa, \lambda) \cap i(\Delta)$. Then $\mu = k(\mu)$, and thus in $M[j_W(G)]$, μ changes cofinality to ω. Therefore, in $V[G]$, $cf(\mu) = \omega$, and, in V, $cf(\mu) \leq \kappa$. The sequence witnessing this belongs to $V \cap ({}^\kappa N)$ and thus, by our assumption on N, belongs already to N. This contradicts the measurability of μ in N.*

Proof. Denote $\lambda = \text{crit}(k)$. Then for some $h \in V$ and $\kappa = \beta_0 < \beta_1 < \ldots < \beta_k$,

$$\lambda = i(h)(\kappa, \beta_1, \ldots, \beta_k)$$

By the definition of k, $\lambda > \kappa$.

We first prove that $\lambda \in i\left(\Delta\right)$. Assume otherwise. We can assume without loss of generality that for every $\xi, \nu_1, \ldots, \nu_k$ below κ, $h\left(\xi, \nu_1, \ldots, \nu_k\right) > \xi$ does not belongs to Δ: this can be assumed by replacing the function h with the function $h' \colon \left[\kappa\right]^{n+1} \to \kappa$ defined as follows: For every $\xi, \eta_1, \ldots, \eta_k$, $h'\left(\xi, \eta_1, \ldots, \eta_k\right)$ equals $h\left(\xi, \eta_1, \ldots, \eta_k\right)$ if $h\left(\xi, \eta_1, \ldots, \eta_k\right) > \xi$ is not measurable in V; and else, $h'\left(\xi, \eta_1, \ldots, \eta_k\right)$ is an arbitrary non-measurable above ξ. By our assumption,

$$i\left(h\right)\left(\kappa, \beta_1, \ldots, \beta_k\right) = i\left(h'\right)\left(\kappa, \beta_1, \ldots, \beta_k\right)$$

so we can replace h with h'. Since λ is regular (as a critical point of an elementary embedding), we can assume, using a similar argument, that each $h\left(\xi, \nu_1, \ldots, \nu_k\right)$ is regular.

We can assume that for every $\xi, \mu_1, \ldots, \nu_k$, there are no elements of Δ in the interval $\left(\xi, h\left(\xi, \nu_1, \ldots, \nu_k\right)\right)$.

Let $f \in V\left[G\right]$ be a function such that $\left[f\right]_W = \lambda$. Then–

$$\left[f\right]_W = \lambda < k(\lambda) = j_W\left(h\right)\left(\kappa, \theta_{\left[f_{\beta_1}\right]_W}, \ldots, \theta_{\left[f_{\beta_k}\right]_W}\right)$$

By the definition of W, there exists $p \in G$ and $r \in H$ such that–

$$p^\frown r \Vdash i(\underset{\sim}{f})(\kappa) < i(h)\big(\kappa, \theta_{i(f_{\beta_1})(\kappa)}, \ldots, \theta_{i(f_{\beta_k})(\kappa)}\big)$$

Recall that, for every $1 \leq i \leq k$, there exists a condition in H forcing that $\theta_{i(f_{\beta_i})(\kappa)} = \beta_i$. Thus by extending r inside H,

$$p^\frown r \Vdash i(\underset{\sim}{f})(\kappa) < i(h)\left(\kappa, \beta_1, \ldots, \beta_k\right)$$

Since there are no measurables of N in the interval $\left(\kappa, i(h)\left(\kappa, \beta_0, \ldots, \beta_k\right)\right]$, we can find $r' \geq^* r$ inside H such that–

$$p \Vdash \exists \alpha < i(h)\left(\kappa, \beta_1, \ldots, \beta_k\right), \; r' \Vdash i(f)(\kappa) < \alpha$$

and since $P = P_\kappa$ is κ-c.c. and $i(h)\left(\kappa, \beta_1, \ldots, \beta_k\right)$ is regular, there exists $\alpha < i(h)\left(\kappa, \beta_1, \ldots, \beta_k\right)$ such that–

$$p^\frown r' \Vdash i(\underset{\sim}{f})(\kappa) < \alpha$$

Now apply k on both sides. By lemma 4.2.17,

$$M\left[j_W(G)\right] \vDash \lambda = \left[f\right]_W < k\left(\alpha\right)$$

but $\alpha < i(h)\left(\kappa, \beta_1, \ldots, \beta_k\right) = \lambda$ and thus $\lambda < k(\alpha) = \alpha < \lambda$, which is a contradiction. $\square$

Remark 4.3.3. *Assume that $P = P_\kappa$ is an iteration of the one point Prikry forcings. A one point Prikry forcing on a measurable α is a forcing, which depends on a normal measure U on α, and is defined as follows: Conditions are of the form A where $A \in U$ or ξ for some ordinal $\xi < \alpha$. The latter kind of condition cannot be extended. A condition of the form A for $A \in U$ can be extended in two ways: A direct extension is a condition B where $B \in U$ and $B \subseteq A$; a non-direct extension is of the form ξ where $\xi \in A$ is an ordinal.*

We argue that in this case, the question whether $(\kappa, crit(k)) \cap i(\Delta) \neq \emptyset$, and, as a result, the value of $crit(k)$, depend of the choice of H:

1. *Denote by μ the first element above κ in $i(\Delta)$. Assume first that H is chosen such that the condition on coordinate μ is a measure one set. In this case, $\mu = crit(k)$. Indeed, $crit(k) < \mu$ cannot hold, since then $(\kappa, crit(k)) \cap i(\Delta) = \emptyset$ which implies, by the last lemma, that $\mu = crit(k)$. And $\mu < crit(k)$ cannot hold since then $k(\mu) = \mu$. Denote by $\mu_0 < \mu$ the one point added below μ in $j_W(G)$. Then H at coordinate μ has a condition which is incompatible with μ_0 (by shrinking the large set and applying a density argument). Thus $\mu = crit(k)$.*

2. *Denote now by μ the least element in $i(\Delta)$, for which H does not specify the one-point element added to it. We argue that $crit(k) = \mu$, even though μ doesn't have to be the least element above κ in $i(\Delta)$.*

   Repeat the proof of the last lemma, and note that the $\leq^*$ forcing in the interval (κ, μ) is trivial, since no condition in this interval can be non-trivially extended. This replaces the assumption that there are no elements of $i(\Delta)$ in the interval $(\kappa, i(h)(\kappa, \beta_1, \ldots, \beta_k))$. Therefore, $\mu = crit(k)$.

Let us deal here with an Easton support iteration P of the Prikry forcings over a set Δ of a measurable length κ. Let U be a normal ultrafilter over κ in V with $\Delta \notin U$. Let $G \subseteq P$ be a generic and W be a normal ultrafilter in $V[G]$ which extends U.

Let $i\colon V \to N$ be an elementary embedding as in theorem 4.1.1, and assume that $W = U_H$ and $k\colon N \to M$ are as in lemma 4.2.14.

In the setting of iteration of Prikry forcings, much more can be said about the embedding $k\colon N \to M$. From remark 4.3.2, it follows that $\text{crit}(k)$ is the least element in $i(\Delta)$ above κ. In particular, by elementarity, $k(\mu) \in j_W(\Delta)$ in M, and thus a Prikry sequence is added to $k(\mu)$ in $j_W(G)$.

Lemma 4.3.4. *Denote $\mu = crit(k)$. Then μ appears in the Prikry sequence of $k(\mu)$.*

Remark 4.3.5. *μ is not necessarily the first element in the Prikry sequence of $k(\mu)$. The initial segment of this Prikry sequence below μ depends on the choice of H. For every finite sequence $t \in [\mu]^{<\omega}$, we can choose $H \subseteq i(P) \setminus \kappa$ such that t is an initial segment of the Prikry sequence of μ. This way, in $M[j_W(G)]$, t will be an initial segment of the Prikry sequence of $k(\mu)$ below μ.*

Proof. Let t be the finite initial segment of the Prikry sequence of $k(\mu)$ below μ, and assume that $\langle \xi, \eta_1, \ldots, \eta_l \rangle \mapsto t(\xi, \eta_1, \ldots, \eta_l)$ is a function in V, such that–

$$t = i\left(\langle \xi, \eta_1, \ldots, \eta_l \rangle \mapsto t(\xi, \eta_1, \ldots, \eta_l)\right)(\kappa, \beta_1, \ldots, \beta_l)$$

for some generators $\beta_1, \ldots, \beta_l$ of i. For every $\xi < \kappa$, let $s(\xi) = \min\{\Delta \setminus (\xi + 1)\}$, so $[\xi \mapsto s(\xi)]_W = \mu$. In $V[G]$, define, for every $\xi < \kappa$,

$$\mu(\xi) = \text{the first element above } t\left(\xi, \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_k}(\xi)}\right) \text{ in the Prikry sequence of } s(\xi)$$

and, if $t\left(\xi, \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_k}(\xi)}\right)$ is not an initial segment of the Prikry sequence of $s(\xi)$, set $\mu(\xi) = 0$.

It suffices to prove that $[\xi \mapsto \mu(\xi)]_W = \mu$.

Assume first that $\eta < \mu$. Work in $N[G]$. Since H is $\leq^*$-generic, it meets an element $q \in i(P) \setminus \kappa$, for which $A^q_\mu \subseteq \mu \setminus (\eta + 1)$. Since $q \in H$, we can assume that t^q_μ is an initial segment of t: Indeed, t, t^q_μ are compatible sequences, since, for any $p \in G$ which forces that $q \in H$ and decides the value of t^q_μ, the condition $k(p^\frown q) = p^\frown k(q)$ belongs to $j_W(G)$, and decides an initial segment, below μ, of the Prikry sequence of $k(\mu)$. By our assumption, this initial segment is contained in t, and $p^\frown k(q)$ forces that every possible extension of it is above η. Thus, in $M[j_W(G)]$, each element in the Prikry sequence of $k(\mu)$ after t is strictly above η.

The argument given in the previous paragraph also shows that for every $q \in H$, t^q_μ is either empty or equals to t: As mentioned, it must be an initial segment of t. Let us argue that if it is proper, then it is empty. Apply the above paragraph for $\eta = \max(t)$. Then by direct extending q inside H, it forces that the element after t^q_μ in the Prikry sequence of μ is strictly above η. By applying $k \colon N \to M$, there exists a condition in $j_W(G)$ which forces that the Prikry sequence of $k(\mu)$ has an initial segment t^q_μ, followed only by elements above η. So t^q_μ cannot be a proper initial segment of t.

Assume now that $\eta < [\xi \mapsto \mu(\xi)]_W$. Write $\eta = [f]_W$ and assume that for every $\xi < \kappa$,

$$f(\xi) < \mu(\xi) < s(\xi)$$

Let $p \in G$ be a condition which forces this. Work in $N[G]$. Take $q \in H$ such that $t^q_\mu = t$. Then $i(p)^\frown q = p^\frown q$ forces that $i(\underset{\sim}{f})(\kappa)$ is below the first element above t in the Prikry sequence of μ. Thus, its value can be decided by taking a direct extension. So, by direct extending q inside H we can assume that–

$$p \Vdash \exists \alpha < \mu,\ q \Vdash i\left(\underset{\sim}{f}\right)(\kappa) < \alpha$$

and thus there exists $\alpha < \mu$ in V, such that–

$$p^\frown q \Vdash i(f)(\kappa) < \alpha$$

Thus, in $M\left[j_W(G)\right]$, $\eta = j_W(f)(\kappa) < k(\alpha) = \alpha < \mu$, as desired. □

In the next subsection we will decompose the embedding k to an iterated ultrapower of N. We now demonstrate the first step in the iteration:

Lemma 4.3.6. *Let $\mu = crit(k)$ and let $U_\mu = \{X \subseteq \mu \colon \mu \in k(X)\} \cap N$. Then $U_\mu \in N$.*

Proof. For every $\xi < \kappa$, denote by W_ξ the measure in $V[G_\xi]$ used to singularize ξ in the Prikry forcing at stage ξ in the iteration. Let $U_\xi = W_\xi \cap V$. We first argue that there exists a set $\mathcal{F} \in N$ of measures on μ, with $|\mathcal{F}| < \mu$, such that, for some $p \in G$ and $q \in H$,

$$p^\frown q \Vdash i\left(\xi \mapsto \underset{\sim}{U}_\xi\right)(\mu) \in \mathcal{F} \tag{4.1}$$

Indeed, let $\underset{\sim}{\alpha}$ be a $j_U(P)$-name for the index of $i\left(\xi \mapsto \underset{\sim}{U}_\xi\right)(\mu)$ in a prescribed well order of the normal measures μ carries in N. Work in $N[G]$. For some $q \in H$, there exists an ordinal β such that $q \Vdash \underset{\sim}{\alpha} = \beta$. Thus, by $\kappa - c.c.$ of the forcing $i(P)_\mu = P_\kappa$, there exist $p \in G$ and a set $S \subseteq 2^{2^\mu}$ of ordinals with $|S| < \mu$, such that $p^\frown q \Vdash \underset{\sim}{\alpha} \in S$. In particular, $p^\frown q$ forces that $i\left(\xi \mapsto \underset{\sim}{U}_\xi\right)(\mu)$ belongs to $\mathcal{F}$, where $\mathcal{F}$ is the set of measures on μ indexed in S.

Now apply k on equation (4.1), and work in $M\left[j_W(G)\right]$. Since $|\mathcal{F}| < \mu$, it follows that there exists a measure $F \in \mathcal{F}$ such that–

$$j_W\left(\xi \mapsto U_\xi\right)\left(k\left(\mu\right)\right) = k\left(F\right)$$

so it suffices to argue that $F = \{X \subseteq \mu \colon \mu \in k(X)\} \cap N$. Fix $X \in F$. Write $X = i(g)(\kappa, \beta_0, \ldots, \beta_k)$. Then–

$$j_W(g)\left(\kappa, \theta_{[f_{\beta_1}]_W}, \ldots, \theta_{[f_{\beta_k}]_W}\right) \in j_W(\xi \mapsto U_\xi)(k(\mu))$$

Recall the function $\xi \mapsto s(\xi) = \min(\Delta \setminus (\xi + 1))$, for which $[\xi \mapsto s(\xi)]_W = k(\mu)$. We can assume that for every $\xi < \kappa$,

$$g\left(\xi, \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_k}(\xi)}\right) \in U_{s(\xi)}$$

and let $p \in G$ be a condition which forces this. Then for strong enough $q \in H$,

$$p^\frown q \Vdash i(g)(\kappa, \beta_1, \ldots, \beta_k) \in i(\xi \mapsto \underset{\sim}{U}_\xi)(\mu)$$

and thus by direct extending q further, we can assume that q forces that the first element after t in the Prikry sequence of μ belongs to $i(g)(\kappa, \beta_1, \ldots, \beta_k) = X$. Thus $k(q) \in j_W(G)$ forces that the first element after t in the Prikry sequence of $k(\mu)$ belongs to $k(X)$. By the previous lemma, it follows that $\mu \in k(X)$, as desired. □

4.3.2 Description of $j_W \restriction_V$

We now generalize the previous subsection, in order to completely decompose $j_W \restriction_V$. We continue to assume that $P = P_\kappa$ is an iterations of Prikry forcings. For technical reasons, we will assume that the measures used in the iteration $P = P_\kappa$ to singularize the measurables in Δ are all simply generated; this is needed only in the proof of claim 4.3.14 which will be presented in the next subsection.

At each stage $\alpha \in \Delta$, let $\underset{\sim}{Q}_\alpha$ be the P_α-name for the Prikry forcing on α, using a simply generated normal measure $\underset{\sim}{W}_\alpha$ on α. Denote $\underset{\sim}{U}_\alpha = \underset{\sim}{W}_\alpha \cap V \in V$. Let $\underset{\sim}{H}_\alpha \subseteq (j_{\underset{\sim}{U}_\alpha}(P_\alpha) \setminus \alpha, \leq^*)$, $\underset{\sim}{H}_\alpha \in V[G_\alpha]$, be $\leq^*$-generic over $M_{\underset{\sim}{U}_\alpha}[G_\alpha]$, such that $\underset{\sim}{W}_\alpha = (\underset{\sim}{U}_\alpha)_{\underset{\sim}{H}_\alpha}$.

Let $G \subseteq P_\kappa$ be generic over V.

Our goal is to prove the following theorem:

Theorem 4.3.1. *Let $H \in V[G]$ be a generic set for $\langle i(P) \setminus \kappa, \leq^* \rangle$ which satisfies $(*)$. Let $W = U_H$ be the corresponding normal measure on κ extending U, and denote its ultrapower embedding $j_W \colon V[G] \to M[j_W(G)] \simeq Ult(V[G], W)$ for some model M. Then $j_W \restriction_V$ factors to the form $j_W \restriction_V = k \circ i$ for some elementary $k \colon N \to M$.*

Moreover, if P is an Easton support iteration, where at each step $\beta \in \Delta$, $\underset{\sim}{Q}_\beta$ is forced to be Prikry forcing with a simply generated normal measure on β, then k is an iterated ultrapower of N by normal measures and $j_W(\kappa) = i(\kappa)$.

This, in contrast to Full-Support and Nonstationary-Support iterations of Prikry forcings, where, assuming $\mathrm{GCH}_{\leq\kappa}$, $j_W \restriction_V$ is an iteration of V by normal measures only.

If all the measures considered, including W, are simply generated, $j_W \restriction_V$ is an iterated ultrapower by normal measures only:

Theorem 4.3.2. *Assume that P is an Easton support iteration, where at each step $\beta \in \Delta$, $\underset{\sim}{Q}_\beta$ is forced to be Prikry forcing with a simply generated normal measure on β. Then for every simply generated measure $W \in V[G]$ on κ, $j_W \restriction_V$ is an iteration of V by normal measures. Moreover, if $U = W \cap V$ then $j_W(\kappa) = j_U(\kappa)$.*

We will prove theorems 4.3.2 and 4.3.1 simultaneously. Assume that $H \in V[G]$ is a generic for $\langle i(P) \setminus \kappa, \leq^* \rangle$ over $N[G]$ with the property $(*)$. In the case where $i = j_U$ and $N = M_U$, any generic for $\langle i(P) \setminus \kappa \rangle, \leq^* \rangle$ is such. Let $W = U_H \in V[G]$ be the corresponding normal measure on κ. Let $j_W \colon V[G] \to M[j_W(G)]$ be the corresponding ultrapower embedding.

Denote by $B \subseteq (\kappa, i(\kappa))$ the set of generators of i. By property $(*)$ of H, for every $\beta \in B$, there exists a function f_β in V such that H forces that $\beta = \theta_{i(f)(\kappa)}$. The mapping $\beta \mapsto f_\beta$ is available in $V[G]$.

Recall the embedding $k \colon N \to M$ defined in lemma 4.2.14:

$$k\left(i(f)\left(\kappa, \beta_1, \ldots, \beta_k\right)\right) = j_W(f)\left(\kappa, \theta_{[f_{\beta_1}]_W}, \ldots, \theta_{[f_{\beta_k}]_W}\right)$$

for every $f \in V$ and $\beta_1, \ldots, \beta_k \in B$. Then k is elementary, $\mathrm{crit}(k) > \kappa$ and $j_W \restriction_V = k \circ i$.

Denote $\kappa^* = i(\kappa)$. Define by induction a linear directed system $\langle \langle M_\alpha \colon \alpha \leq \kappa^* \rangle, \langle j_{\alpha,\beta} \colon \alpha < \beta \leq \kappa^* \rangle \rangle$ such that:

1. $M_0 = N$, $j_0 = i$.

2. **Successor Step:** Assume that $\alpha < \kappa^*$ and M_α has been defined. We will define an elementary embedding $k_\alpha \colon M_\alpha \to M$, such that $j_W \restriction_V = k_\alpha \circ j_\alpha$. We denote $\mu_\alpha = \mathrm{crit}(k_\alpha)$ and define–

$$U_{\mu_\alpha} = \{X \subseteq \mu_\alpha \colon \mu_\alpha \in k_\alpha(X)\} \cap M_\alpha$$

We will prove that $U_{\mu_\alpha} \in M_\alpha$ and take $M_{\alpha+1} \simeq \mathrm{Ult}\left(M_\alpha, U_{\mu_\alpha}\right)$. We also take $j_{\alpha,\alpha+1}\colon M_\alpha \to M_{\alpha+1}$ to be the ultrapower embedding $j^{M_\alpha}_{U_{\mu_\alpha}}$, and $j_{\alpha+1} = j_{\alpha,\alpha+1} \circ j_\alpha$.

3. **Limit Step:** For every limit $\alpha \leq \kappa^*$, the system $\langle M_\beta \colon \beta < \alpha\rangle, \langle j_{\beta,\gamma}\colon \beta < \gamma < \alpha\rangle$ is linearly directed, and we take direct limit to form the model M_α and the embedding $j_\alpha\colon V \to M_\alpha$.

For every $\alpha < \kappa^*$, define $k_\alpha\colon M_\alpha \to M$ as follows:

$$k_\alpha\left(j_\alpha\left(f\right)\left(\kappa, j_{0,\alpha}(\beta_1), \ldots, j_{0,\alpha}\left(\beta_l\right), \mu_{\alpha_1}, \ldots, \mu_{\alpha_k}\right)\right) = j_W\left(f\right)\left(\kappa, \theta_{[f_{\beta_1}]_W}, \ldots, \theta_{[f_{\beta_l}]_W}, \mu_{\alpha_1}, \ldots, \mu_{\alpha_k}\right)$$

for every $f \in V$, $\beta_1, \ldots, \beta_l$ generators of i and $\alpha_1 < \ldots < \alpha_k < \alpha$.

Our goal is to prove by induction on $\alpha < \kappa^*$ the following properties:

(A) $k_\alpha\colon M_\alpha \to M$ is an elementary embedding, and $j_W \restriction_V = k_\alpha \circ j_\alpha$.

(B) μ_α is measurable in M_α. Moreover, it is the least measurable in $j_\alpha\left(\Delta\right)$, which is greater or equal to $\sup\{\mu_\beta \colon \beta < \alpha\}$, and whose cofinality is above κ in V.

(C) μ_{μ_α} appears in the Prikry sequence of $k_\alpha\left(\mu_\alpha\right)$.

(D) Let U_{μ_α} be defined in $V\left[G\right]$ as above. Then $U_{\mu_\alpha} \in M_\alpha$ is a normal measure which concentrates on $\mu_\alpha \setminus j_\alpha\left(\Delta\right)$. Moreover,

$$k_\alpha\left(U_{\mu_\alpha}\right) = j_W\left(\delta \mapsto U_\delta\right)\left(k_\alpha\left(\mu_\alpha\right)\right)$$

where, for every $\delta \in \Delta$, $U_\delta = W_\delta \cap V$, for W_δ which is the measure used in the Prikry forcing at stage δ in the iteration P.

After that, we will prove in lemma 4.3.20, that $k_{\kappa^*}\colon M_{\kappa^*} \to M$ is the identity, and thus $j_W \restriction_V = j_{\kappa^*}$. This will conclude the proof of theorems 4.3.2 and 4.3.1.

Remark 4.3.7. *We remark that k_α is well defined is the sense that there is no $\alpha' < \alpha$ and generator β of i, for which $j_{0,\alpha}(\beta) = \mu_{\alpha'}$. Indeed, assume otherwise. Note that $\mu_{\alpha'} = j_{0,\alpha}\left(\beta\right) \geq j_{0,\alpha'}(\beta)$. Strict inequality is not possible here, since if $j_{0,\alpha'}(\beta) < \mu_{\alpha'}$ then $j_{0,\alpha'}(\beta) = j_{0,\alpha}(\beta) = \mu_{\alpha'}$, which is a contradiction. Thus, $j_{0,\alpha'}\left(\beta\right) = \mu_{\alpha'}$ (which is, by itself, possible for $\alpha' < \alpha$ - see remark 4.3.8), but then, applying $j_{\alpha',\alpha}$ on both sides, we get–*

$$j_{0,\alpha}(\beta) = j_{\alpha',\alpha}\left(\mu_{\alpha'}\right) > \mu_{\alpha'}$$

where the last inequality follows since $\mu_{\alpha'} = crit\left(j_{\alpha',\alpha}\right)$.

Remark 4.3.8. *It is possible that a generator β of i is measurable in N and belongs to $i(\Delta)$. In this case, there exists $\alpha < \kappa^*$ such that $\mu_\alpha = \beta = j_{0,\alpha}(\beta)$. Such β will appear as an element in the Prikry sequence of $k_\alpha(\beta) \in j_W(\Delta)$, which also has the form $\theta_{[f_\beta]_W}$.*

Properties $(A) - (D)$ of k_α, presented above, will be proved by induction on $\alpha < \kappa^*$. The proof of the inductive step at stage $\alpha < \kappa^*$ will be carried out in subsection 4.3.4, using the tools presented in [9] and [15]. Fixing $\alpha < \kappa^*$, we can assume by induction that $k_{\alpha'} \colon M_{\alpha'} \to M$ and $\mu_{\alpha'}, U_{\mu_{\alpha'}}$, for $\alpha' < \alpha$, satisfy properties $(A) - (D)$. Denote by $t_{\alpha'}$ the initial segment of the Prikry sequence of $k_{\alpha'}(\mu_{\alpha'})$ below $\mu_{\alpha'}$.

Definition 4.3.9. *Fix $\alpha < \kappa^*$ and a sequence of generators $\langle \beta_1, \ldots, \beta_l \rangle$ for i. An increasing sequence $\langle \alpha_1, \ldots, \alpha_k \rangle$ below α is called a $\langle \beta_1, \ldots, \beta_l \rangle$-nice sequence if there are functions $g_1, \ldots, g_k, t_1, \ldots, t_k$ in V, such that–*

$$\mu_{\alpha_1} = j_{\alpha_1}(g_1)(\kappa, j_{0,\alpha_1}(\beta_1), \ldots, j_{0,\alpha_1}(\beta_l))$$

$$t_{\alpha_1} = j_{\alpha_1}(t_{\alpha_1})(\kappa, j_{0,\alpha_1}(\beta_1), \ldots, j_{0,\alpha_1}(\beta_l))$$

$$U_{\mu_{\alpha_1}} = j_{\alpha_1}(F_1)(\kappa, j_{0,\alpha_1}(\beta_1), \ldots, j_{0,\alpha_1}(\beta_l))$$

and, for every $1 \le i < k$,

$$\mu_{\alpha_{i+1}} = j_{\alpha_{i+1}}(g_{i+1})(\kappa, j_{0,\alpha_1}(\beta_1), \ldots, j_{0,\alpha_1}(\beta_l), \mu_{\alpha_1}, \ldots, \mu_{\alpha_i})$$

$$t_{\alpha_{i+1}} = j_{\alpha_{i+1}}(t_{i+1})(\kappa, j_{0,\alpha_1}(\beta_1), \ldots, j_{0,\alpha_1}(\beta_l), \mu_{\alpha_1}, \ldots, \mu_{\alpha_i})$$

$$U_{\mu_{\alpha_{i+1}}} = j_{\alpha_{i+1}}(F_{i+1})(\kappa, j_{0,\alpha_1}(\beta_1), \ldots, j_{0,\alpha_1}(\beta_l), \mu_{\alpha_1}, \ldots, \mu_{\alpha_i})$$

Fix now $\alpha < \kappa^*$. Assume by induction that properties $(A) - (D)$ above hold for every $\alpha' < \alpha$. Fix also a sequence of generators $\langle \beta_1, \ldots, \beta_l \rangle$ for i, and a $\langle \beta_1, \ldots, \beta_l \rangle$-nice sequence $\langle \alpha_1, \ldots, \alpha_k \rangle$ below α. We define, in $V[G]$, functions which can be used to represent μ_{α_i}, $t_{\alpha_i}, U_{\alpha_i}$. Assume that μ_{α_i} is the n_i-th element in the Prikry sequence of $k_{\alpha_i}(\mu_{\alpha_i})$.

First, set–

$$\mu_{\alpha_1}(\xi) = \text{the } n_1\text{-th element in the Prikry sequence of } g_1(\xi, \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)})$$

By induction, define, for every $i < k$,

$$\mu_{\alpha_{i+1}}(\xi) = \text{the } n_{i+1}\text{-th element in the Prikry sequence of}$$
$$g_{i+1}(\xi, \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_i}(\xi))$$

and $U_{\mu_{\alpha_i}}(\xi) = W_{\mu_{\alpha_i}(\xi)} \cap V$. Here, given $\delta \in \Delta$, W_δ is the measure on δ used in the Prikry forcing which was applied at stage δ in the iteration.

Claim 4.3.10. *$[\xi \mapsto \mu_{\alpha_i}(\xi)]_W = \mu_{\alpha_i}$ and $\left[\xi \mapsto U_{\mu_{\alpha_i}(\xi)}\right]_W = k_{\alpha_i}\left(U_{\mu_{\alpha_i}}\right)$.*

Proof. We begin by proving that $[\xi \mapsto \mu_{\alpha_i}(\xi)]_W = \mu_{\alpha_i}$. We present the argument for $i = 1$. Higher values of $i \le k$ are proved similarly, using induction. Recall that–

$$\mu_{\alpha_1} = j_{\alpha_1}(g_1)(\kappa, j_{0,\alpha_1}(\beta_1), \ldots, j_{0,\alpha_1}(\beta_l))$$

and by applying k_{α_1} on both sides,

$$k_{\alpha_1}(\mu_{\alpha_1}) = j_W(g_1)(\kappa, \theta_{[f_{\beta_1}(\xi)]_W}, \ldots, \theta_{[f_{\beta_l}(\xi)]_W})$$

By induction, μ_{α_1} is the n_1-th element in the Prikry sequence of $k_{\alpha_1}(\mu_{\alpha_1})$, and thus it is represented as the n_1-th element in the Prikry sequence of $g_1\left(\xi, \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}\right)$.

As for $\left[\xi \mapsto U_{\mu_{\alpha_i}(\xi)}\right]_W = k_{\alpha_i}\left(U_{\mu_{\alpha_i}}\right)$, this follows since, by induction,

$$k_{\alpha_i}\left(U_{\mu_{\alpha_i}}\right) = j_W(\delta \mapsto U_\delta)(k_{\alpha_i}(\mu_{\alpha_i}))$$

□

Let us argue that $k_\alpha \colon M_\alpha \to M$ is elementary.

Lemma 4.3.11. *$k_\alpha \colon M_\alpha \to M$ is elementary.*

Proof. Assume that $x, y \in M_\alpha$, and let us prove, for example, that $x \in y$ if and only if $k(x) \in k(y)$. Let $f, g \in V$, $\beta_1, \ldots, \beta_l$ and $\alpha_1 < \ldots < \alpha_k < \alpha$ be such that–

$$x = j_\alpha(f)(\kappa, j_{0,\alpha}(\beta_1), \ldots, j_{0,\alpha}(\beta_l), \mu_{\alpha_1}, \ldots, \mu_{\alpha_k})\ ,\ y = j_\alpha(g)(\kappa, j_{0,\alpha}(\beta_1), \ldots, j_{0,\alpha}(\beta_l), \mu_{\alpha_1}, \ldots, \mu_{\alpha_k})$$

Assume that $\alpha = \alpha' + 1$ is successor (the limit case is simpler). For simplicity, we assume also that $\alpha_k = \alpha'$. Then $x \in y$ if and only if–

$$\mu_{\alpha'} \in j_{\alpha',\alpha}\left(\{\xi < \mu_{\alpha'} \colon j_{\alpha'}(f)\left(\kappa, j_{0,\alpha'}(\beta_1), \ldots, j_{0,\alpha'}(\beta_l), \mu_{\alpha_1}, \ldots, \mu_{\alpha_{k-1}}, \xi\right) \in j_{\alpha'}(g)\left(\kappa, j_{0,\alpha'}(\beta_1), \ldots, j_{0,\alpha'}(\beta_l), \mu_{\alpha_1}, \ldots, \mu_{\alpha_{k-1}}, \xi\right)\}\right)$$

which is equivalent to–

$$\{\xi < \mu_{\alpha'} \colon j_{\alpha'}(f)\left(\kappa, j_{0,\alpha'}(\beta_1), \ldots, j_{0,\alpha'}(\beta_l), \mu_{\alpha_1}, \ldots, \mu_{\alpha_{k-1}}, \xi\right) \in$$
$$j_{\alpha'}(g)\left(\kappa, j_{0,\alpha'}(\beta_1), \ldots, j_{0,\alpha'}(\beta_l), \mu_{\alpha_1}, \ldots, \mu_{\alpha_{k-1}}, \xi\right)\} \in U_{\mu_{\alpha'}}$$

which, by the definition of $U_{\mu_{\alpha'}}$, is equivalent to–

$$\mu_{\alpha'} \in k_{\alpha'}\left(\{\xi < \mu_{\alpha'} \colon j_{\alpha'}(f)\left(\kappa, j_{0,\alpha'}(\beta_1), \ldots, j_{0,\alpha'}(\beta_l), \mu_{\alpha_1}, \ldots, \mu_{\alpha_{k-1}}, \xi\right) \in\right.$$
$$\left. j_{\alpha'}(g)\left(\kappa, j_{0,\alpha'}(\beta_1), \ldots, j_{0,\alpha'}(\beta_l), \mu_{\alpha_1}, \ldots, \mu_{\alpha_{k-1}}, \xi\right)\}\right)$$

namely $k_\alpha(x) \in k_\alpha(y)$.

□

Let us describe now the main ideas behind the proof that $\mu_\alpha = \mathrm{crit}\,(k_\alpha)$ is measurable in M_α. Note that this is not trivial since $k_\alpha \colon M_\alpha \to M$ is not definable in M_α. The full argument will be presented in lemma 4.3.17, but will require a technical theorem (theorem 4.3.3). Mainly we would like to follow the methods developed in [9] and [15], which deal with nonstationary and full support iterations of Prikry forcings, respectively.

We consider the function $f \in V[G]$, for which $\mu_\alpha = [f]_W$. We will prove that if μ_α is not measurable in M_α, then $\mu_\alpha = [f]_W \in \mathrm{Im}\,(k_\alpha)$, contradicting the fact that $\mu_\alpha = \mathrm{crit}\,(k_\alpha)$. For that, we first fix a function $h \in V$ such that, for some sequence $\beta_1, \ldots, \beta_l$ of generators of i, and for some nice sequence $\langle \alpha_1, \ldots, \alpha_k \rangle$ below α,

$$\mu_\alpha = j_\alpha(h)\left(\kappa, j_{0,\alpha}(\beta_1), \ldots, j_{0,\alpha}(\beta_l), \mu_{\alpha_1}, \ldots, \mu_{\alpha_k}\right)$$

since $\mu_\alpha = \mathrm{crit}\,(k_\alpha)$, we can assume that for every $\xi < \kappa$,

$$f(\xi) < h\left(\xi, \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right)$$

Pick a condition $p \in G$ which forces this. For every $\xi < \kappa$, $\vec{\eta} = \langle \eta_1, \ldots, \eta_l \rangle$ and $\vec{\nu} = \langle \nu_1, \ldots, \nu_k \rangle$, denote–

$$e\left(\xi, \vec{\eta}, \vec{\nu}\right) = \{r \in P \setminus \nu_k \colon \text{ there exists a bounded subset } A \subseteq h\left(\xi, \vec{\eta}, \vec{\nu}\right) \text{ such that } r \Vdash \underset{\sim}{f}(\xi) \in A\}$$

This set is $\leq^*$-dense open above conditions which extend p and force that–

$$\langle \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi) \rangle = \langle \vec{\eta}, \vec{\nu} \rangle \tag{4.2}$$

We would like to follow [9] and [15], and construct a condition $p^* \in G$ above p, such that, very roughly[5], for every $\xi, \vec{\eta}, \vec{\nu}$ as above, and for every extension r of p^* which forces (4.2),

$$r\restriction_{\nu_k} \Vdash r\backslash_{\nu_k} \in e(\xi, \vec{\eta}, \vec{\nu})$$

Essentially, such p^* will have the following property: every extension r of it which forces that equation (4.2) holds, forces also that $f(\xi)$ belongs to a bounded subset $A(\xi, \vec{\eta}, \vec{\nu}) \subseteq h(\xi, \vec{\eta}, \vec{\nu})$ (which depends only on p^* and $\langle \xi, \vec{\eta}, \vec{\nu} \rangle$, and not on the choice of the extension of p^* which forces (4.2)). In [9] and [15] the construction of such p^* was done by a Fusion argument which allows, in a sense, to absorb a lot of data into a single direct extension p^* of p. Such a method is not available in the Easton support iteration. We bypass this problem by constructing, for every sequence $\langle \xi, \eta_1, \ldots, \eta_l \rangle$, a system of non-direct extensions of p,

$$\langle p(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_k) : \nu_1 < \ldots < \nu_k < \kappa \rangle$$

and sets–

$$\langle A(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_k) : \nu_1 < \ldots < \nu_k < \kappa \rangle$$

such that the following properties hold:

1. If $p(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_k)$ forces (4.2), then it also forces that $\underset{\sim}{f}(\xi) \in A(\xi, \vec{\eta}, \vec{\nu})$, which is a bounded subset of $h(\xi, \vec{\eta}, \vec{\nu})$.
2. For a set of ξ-s in W, $p\left(\xi, \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right)$ belongs to G.

This suffices, since, by combining the above properties,

$$V[G] \vDash \{\xi < \kappa \colon f(\xi) \in A\left(\xi, \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right)\} \in W$$

and thus, in $M[j_W(G)]$,

$$\begin{aligned}\mu_\alpha = [f]_W \in & \left[\xi \mapsto A\left(\xi, \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right)\right]_W \\ &= k_\alpha\left(\, j_\alpha\left(\langle \xi, \vec{\eta}, \vec{\nu} \rangle \mapsto A(\xi, \vec{\eta}, \vec{\nu})\right)(\kappa, \beta_1, \ldots, \beta_l, \mu_1, \ldots, \mu_k)\,\right) \subseteq \operatorname{Im}(k_\alpha)\end{aligned}$$

where the last inclusion follows since $j_\alpha\left(\langle \xi, \vec{\eta}, \vec{\nu} \rangle \mapsto A(\xi, \vec{\eta}, \vec{\nu})\right)(\kappa, \beta_1, \ldots, \beta_l, \mu_1, \ldots, \mu_k)$ is a bounded subset of $\mu_\alpha = j_\alpha(h)(\kappa, \beta_1, \ldots, \beta_l, \mu_{\alpha_1}, \ldots, \mu_{\alpha_k})$.

We will complete the missing details in the proof in lemma 4.3.17. Before that, we present the proof of theorem 4.3.3.

[5] We omitted some of the details in the version described here, for sake of simplicity.

4.3.3 Theorem 4.3.3 and its proof

We devote this subsection to the proof of the following theorem:

Theorem 4.3.3. *Let $p \in G$ be a condition. Assume that for every increasing sequence $\langle \xi, \nu_1, \ldots, \nu_k \rangle$, and for every $\vec{\eta} = \langle \eta_1, \ldots, \eta_l \rangle$ above ξ, the set–*

$$e(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_k) \subseteq P \setminus \nu_k$$

is $\leq^*$ dense open above conditions in $P \setminus \nu_k$ which force that–

$$\langle \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_k \rangle = \langle \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi) \rangle$$

Then there are $s < \omega$, a new sequence of generators $\beta'_l, \ldots, \beta'_s$ of i which contains $\beta_1, \ldots, \beta_l$, and a system of extensions of p,

$$\langle p(\xi, \eta_1, \ldots, \eta_s, \nu_1, \ldots, \nu_k) : \eta_1, \ldots, \eta_s < \kappa, \nu_1 < \ldots < \nu_k < \kappa \rangle$$

with the following properties:

1. *There exists a set of ξ-s in W for which–*

$$\begin{aligned} &p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \restriction_{\mu_{\alpha_k}(\xi)} \Vdash \\ &p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \setminus \mu_{\alpha_k}(\xi) \in \\ &e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \end{aligned}$$

2. *There exists a set of ξ-s in W for which–*

$$p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \in G$$

(Intuitively, for the majority of values of $\langle \xi, \eta_1, \ldots, \eta_s, \nu_1, \ldots, \nu_k \rangle$, the condition $p(\xi, \eta_1, \ldots, \eta_s, \nu_1, \ldots, \nu_k)$ which we will construct, forces that–

$$\langle \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi) \rangle = \langle \eta_1, \ldots, \eta_s, \nu_1, \ldots, \nu_k \rangle$$

and its final segment belongs to $e(\xi, \eta_1, \ldots, \eta_s, \nu_1, \ldots, \nu_k)$).

Remark 4.3.12. *When we extend a sequence of generators $\langle \beta_1, \ldots, \beta_l \rangle$ to a sequence $\langle \beta'_1, \ldots, \beta'_s \rangle$ we will naturally identify the set $e(\xi, \eta_1, \ldots, \eta_l)$, with–*

$$e'(\xi, \eta_1, \ldots, \eta_s) = e(\xi, \eta_{i_1}, \ldots, \eta_{i_l})$$

where i_j is the index for which $\beta'_{i_j} = \beta_j$, for every $1 \leq j \leq l$.

Similarly, whenever a function $g \in V$ is given, whose variables are $\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_k$, we abuse the notation and denote $g(\xi, \eta_1, \ldots, \eta_s, \nu_1, \ldots, \nu_k)$ to mean $g(\xi, \eta_{i_1}, \ldots, \eta_{i_l}, \nu_1, \ldots, \nu_k)$.

The proof of theorem 4.3.3 goes by generalizing the given sets $e(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_k)$:

Definition 4.3.4. *For every $\eta_1, \ldots, \eta_l < \kappa$, $1 \leq i \leq k$ and an increasing sequence $\langle \xi, \nu_1, \ldots, \nu_i \rangle$, we define a set $e(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i) \subseteq P \setminus \nu_i$.*
For $i = k$ this is the set $e(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_k)$ given in the formulation of the theorem.
Assume that $1 \leq i < k$. Work by recursion. Assume that for every $\nu < g_{i+1}(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i)$, the set $e(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i, \nu)$ is defined. Denote $g_{i+1} = g_{i+1}(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i)$. Let us define the set $e(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i)$, as follows: A condition $q \in P \setminus \nu_i$ belongs to $e(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i)$ if and only if the following properties hold:

1. *(A technical requirement) $q \restriction_{g_{i+1}}$ decides the statements–*

   $$F_{i+1}(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_j) = \underset{\sim}{W}_{g_{i+1}} \cap V \ , \ t^q_{g_{i+1}} = t_{i+1}(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i)$$

   Also, if $q \restriction_{g_{i+1}}$ decides that $t^q_{g_{i+1}} \neq t_{i+1}(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i)$, it also decides whether one of the sequences is an initial segment of the other, and if so, which one it is. Finally, if it forces that $t^q_{g_{i+1}}$ is a strict initial segment of $t_{i+1}(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i)$, it also forces that $A^q_{g_{i+1}} \subseteq g_{i+1} \setminus \max(t_{i+1}(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i))$.

2. *(The essential requirement) If both statements in the technical requirement are decided positively, there exists a sequence–*

   $$\langle \ q(\nu) : \nu < g_{i+1}(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i) \ \rangle$$

   such that, for every $\nu < g_{i+1}(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i)$ above ν_i, $q(\nu) \in P \setminus \nu$ extends $q \setminus \nu$, and–

   $$q \Vdash \text{ if } \underset{\sim}{\mu}_{\alpha_{i+1}}(\xi) = \nu, \text{ then } q(\nu) \in G \setminus \nu \text{ and } q(\nu) \in e(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i, \nu)$$

Similarly, given $\langle \xi, \eta_1, \ldots, \eta_l \rangle$*, define* $e(\xi, \eta_1, \ldots, \eta_l)$ *to be the set of conditions* $q \in P \setminus \xi$ *which decide whether* $F_1(\xi, \eta_1, \ldots, \eta_l) = W_{g_1(\xi,\eta_1,\ldots,\eta_l)} \cap V$*,* $t_1(\xi, \eta_1, \ldots, \eta_l) = t^q_{g_1(\xi,\eta_1,\ldots,\eta_l)}$*, and, assuming that it is decided positively, have a system of extensions–*

$$\langle q(\nu) : \nu < g_1(\xi, \eta_1, \ldots, \eta_l) \rangle$$

such that, for every $\nu < g_1(\xi, \eta_1, \ldots, \eta_l)$*,* $q(\nu) \in P \setminus \nu$*, and–*

$$q \Vdash \textit{if } \underset{\sim}{\mu}_{\alpha_1}(\xi) = \nu \textit{ then } q(\nu) \in G \setminus \nu \textit{ and } q(\nu) \in e(\xi, \eta_1, \ldots, \eta_l, \nu)$$

If it is decided negatively, then $q \restriction_{g_1}$ *knows how to compare* $t^q_{g_1}$ *and* $t_1(\xi, \eta_1, \ldots, \eta_l)$ *as in the second point above.*

By induction, we will argue that for every $i \leq k$ and $\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i$, the set $e(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i) \subseteq P \setminus \nu_i$ is $\leq^*$-dense open above conditions $q \in P \setminus \nu_i$ for which–

$$q \Vdash \langle \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}, \underset{\sim}{\mu}_{\alpha_1}(\xi), \ldots, \underset{\sim}{\mu}_{\alpha_i}(\xi) \rangle = \langle \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i \rangle, \text{ and for}$$
$$\text{every } 1 \leq j \leq i,\ F_{j+1}(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_j) = \underset{\sim}{W}_{g_{j+1}(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_j)} \text{ and}$$
$$t_{j+1}(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_j) = t^q_{g_{j+1}(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_i)}$$

The induction will be inverse: The basis, for $i = k$, is true, as it is known that the set $e(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_k) \subseteq P \setminus \nu_k$ is $\leq^*$ dense–open above conditions $q \in P \setminus \nu_k$ which force that–

$$\langle \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}, \underset{\sim}{\mu}_{\alpha_1}(\xi), \ldots, \underset{\sim}{\mu}_{\alpha_k}(\xi) \rangle = \langle \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_k \rangle$$

The inductive step is given in the following lemma:

Lemma 4.3.13. *Fix* $\eta_1, \ldots, \eta_l < \kappa$*,* $1 \leq i < k$ *and an increasing sequence* $\langle \xi, \nu_1, \ldots, \nu_i \rangle$*. Denote* $g_{i+1} = g_{i+1}(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i)$*. Assume that for every* $\nu_{i+1} \in (\nu_i, g_{i+1})$*, the set–*

$$e(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i, \nu_{i+1}) \subseteq P \setminus \nu_{i+1}$$

is $\leq^*$*-dense open above conditions* $q \in P \setminus \nu_{i+1}$ *for which–*

$$q \Vdash \langle \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}, \underset{\sim}{\mu}_{\alpha_1}(\xi), \ldots, \underset{\sim}{\mu}_{\alpha_i}(\xi), \underset{\sim}{\mu}_{\alpha_{i+1}}(\xi) \rangle = \langle \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i, \nu \rangle, \textit{ and for}$$
$$\textit{every } 1 \leq j \leq i+1,\ F_{j+1}(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_j) = \underset{\sim}{W}_{g_{j+1}(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_j)} \textit{ and}$$
$$t_{j+1}(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_j) = t^q_{g_{j+1}(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_i)}$$

then $e(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_i)$ *is* $\leq^*$*-dense open above conditions* $q\in P\setminus\nu_i$ *for which–*

$$q\Vdash\langle\theta_{f_{\beta_1}(\xi)},\ldots,\theta_{f_{\beta_l}(\xi)},\underset{\sim}{\mu}_{\alpha_1}(\xi),\ldots,\underset{\sim}{\mu}_{\alpha_i}(\xi)\rangle=\langle\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_i\rangle,\ \textit{and for}$$
$$\textit{every } 1\leq j\leq i,\ F_{j+1}(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_j)=\underset{\sim}{W}_{g_{j+1}(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_j)}\ \textit{and}$$
$$t_{j+1}(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_j)=t^q_{g_{j+1}(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_i)}$$

Proof. Let $q\in P\setminus\nu_i$ be a condition which forces that–

$$\theta_{f_{\beta_1}(\xi)},\ldots,\theta_{f_{\beta_l}(\xi)},\underset{\sim}{\mu}_1(\xi),\ldots,\underset{\sim}{\mu}_i(\xi)\rangle=\langle\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_i\rangle$$
$$\text{and for every } 1\leq j\leq i,\ F_{j+1}(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_j)=\underset{\sim}{W}_{g_{j+1}(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_j)}$$
$$\text{and } t_{j+1}(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_j)=t^q_{g_{j+1}(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_i)}$$

Denote–

$$g=g_{i+1}(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_i)$$
$$U_g=F_{i+1}(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_i)$$
$$t=t_{i+1}(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_i)$$

Assume that $q\restriction_g$ forces that–

$$\underset{\sim}{W}_g\cap V=U_g,\ t=\underset{\sim}{t}^{\,q}_g$$

(if not, we are done since $q\in e(\xi,\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_i)$). Denote $n=\mathrm{lh}(t)$. We will now apply the following claim:

Claim 4.3.14. *Assume that* $p\in G$ *is a condition,* $n<\omega$ *and* $g\in\Delta$ *is measurable in* V*. Assume that* U_g *is a normal measure on* g *in* V*,* t *is a finite sequence below* g *of length* n*, and–*

$$p\Vdash\underset{\sim}{t}^{\,q}_g=t,\ \underset{\sim}{W}_g\cap V=U_g$$

For every $\nu<g$*, assume that* $e(\nu)\subseteq P\setminus\nu$ *is a* P_ν*-name for a subset of* $P\setminus\nu$*, which is* $\leq^*$ *dense-open above conditions which force that* ν *is the* $(n+1)$*-th element in the Prikry sequence of* g*. Then there exists a direct extension* $p^*\geq^* p$ *and a sequence* $\langle p(\nu):\nu<g\rangle$*, such that, for every* $\nu<g$*,*

$$p^*\Vdash\textit{if } \nu \textit{ appears after } t \textit{ in the Prikry sequence of } g,\ \textit{then } p(\nu)\in(G\setminus\nu)\cap e(\nu)$$
$$\textit{and } p^*\restriction_\nu\Vdash p(\nu)\geq^* p^*\restriction_{[\nu,g)}{}^\frown\langle t^\frown\langle\nu\rangle,\underset{\sim}{A}^{p^*}_g\setminus\nu\rangle^\frown p^*\setminus(g+1).$$

Proof. For every $\nu < g$, consider the set–

$$d(\nu) = \{r \in P \restriction_{[\nu,g)} : r \parallel \nu \in \underset{\sim}{A}_g^p, \text{ and if } r \Vdash \nu \in \underset{\sim}{A}_g^p \text{ then}$$
$$r \Vdash \exists s \geq^* \langle t^\frown\langle\nu\rangle, \underset{\sim}{A}_g^p \setminus \nu\rangle^\frown p \setminus (g+1)\,,\ r^\frown s \in e(\nu)\}$$

Then $d(\nu) \subseteq P \restriction_{[\nu,g)}$ is $\leq^*$-dense open above $p \restriction_{[\nu,g)}$. Let H_g be the P_g-name, forced by $p \restriction_g$, to be the $\leq^*$-generic subset of $j_{U_g}(P_g) \setminus g$, for which–

$$\underset{\sim}{W}_g = (U_g)_{\underset{\sim}{H}_g}$$

(such a generic exists since W_g is simply generated). Let $\underset{\sim}{q} \in \mathrm{Ult}(V, U_g)$ be a P_g-name, forced by p to be a condition in $[\nu \mapsto d(\nu)]_{U_g} \cap \underset{\sim}{H}_g$. Let $\nu \mapsto \underset{\sim}{q}(\nu) \in P \restriction_{[\nu,g)}$ be a function in V such that $[\nu \mapsto \underset{\sim}{q}(\nu)]_{U_g} = \underset{\sim}{q}$. Then we can assume that for a set of ν-s in U_g,

$$p \restriction_\nu \Vdash \underset{\sim}{q}(\nu) \in d(\nu) \tag{4.3}$$

and, by lemma 4.2.17, $p \restriction_g$ forces that there exists a set $\underset{\sim}{C} \in W_g$, such that for every $\nu \in C$,

$$p \restriction_\nu {}^\frown \underset{\sim}{q}(\nu) \in \underset{\sim}{G} \restriction_g$$

By shrinking C if necessary, we can assume that every $\nu \in C$ also satisfies equation (4.3). Now let us define the extension $p^* \geq^* p$, and, for every $\nu < g$, the condition $p(\nu) \in P \setminus \nu$. First, set–

$$p^* \restriction_g = p \restriction_g$$

and, in $V^{P\restriction_\nu}$, set–

$$p(\nu) \restriction_g = \underset{\sim}{q}(\nu)$$

Work in an arbitrary generic extension for $P \restriction_g$, where $p^* \restriction_g$ belongs. For every $\nu \in C \cap A_g^p$ (which thus satisfies $p \restriction_\nu {}^\frown \underset{\sim}{q}(\nu) \in G \restriction_g$), there exists $s(\nu) \in P \setminus g$, $s(\nu) \geq^* \langle t^\frown\langle\nu\rangle, \underset{\sim}{A}_g^p \setminus \nu\rangle^\frown q \setminus (g+1)$, such that $p(\nu) \restriction_g {}^\frown s(\nu) \in e(\nu)$. Set–

$$p^*(g) = \langle \underset{\sim}{t}_g^p, \underset{\sim}{A}_g^p \cap C \cap \left(\triangle_{\nu < g,\ \nu \in C \cap A_g^p} \underset{\sim}{A}_g^{s(\nu)} \right) \rangle$$

(the definition above is carried in $V[G \restriction_g]$, so $\underset{\sim}{C}$ is available there).

Let $p^* \setminus (g+1) = s(\underset{\sim}{\nu})$, where $\underset{\sim}{\nu}$ is the $(n+1)$-th element in the Prikry sequence of g. Finally, let–

$$p(\nu) \setminus g = \langle t^\frown\langle\nu\rangle, A_g^{p^*} \setminus \nu\rangle^\frown p^* \setminus (g+1)$$

where the above definition is possible if $p\restriction_{\nu}{}^{\frown}p(\nu)\restriction_{g}\Vdash \nu\in \underset{\sim}{A}_{g}^{p^{*}}$; if not, let $p(\nu)\setminus g$ be arbitrary.

This completes the definition of $q^{*}\geq^{*}q$ and $\langle p(\nu):\nu<g\rangle$. Let us prove that for every $\nu<g$,

$$p^{*}\Vdash\text{if } \nu \text{ appears after } t \text{ in the Prikry sequence of } g,\text{ then } p(\nu)\in(G\setminus\nu)\cap e(\nu)$$
$$\text{and } p^{*}\restriction_{\nu}\Vdash p(\nu)\geq^{*}p^{*}\restriction_{[\nu,g)}{}^{\frown}\langle t^{\frown}\langle\nu\rangle,\underset{\sim}{A}_{g}^{p^{*}}\setminus\nu\rangle^{\frown}p^{*}\setminus(g+1).$$

Fix $\nu<g$ and let G be a generic set for P which includes p^{*}, such that, in $V[G]$, ν appears after t in the Prikry sequence of g. In particular, $\nu\in C$ and thus $q(\nu)\in G\restriction_{[\nu,g)}$. By the definition of $p(\nu)$, and since $p^{*}\in G, q(\nu)\in G\restriction_{[\nu,g)}$, it follows that $p(\nu)\in G\setminus\nu$, as desired.

$\square$ of claim 4.3.14.

Apply claim 4.3.14 with respect to the set $e(\xi,\eta_{1},\ldots,\eta_{l},\nu_{1},\ldots,\nu_{i},\nu)\subseteq P\setminus\nu$ (recall that $\xi,\eta_{1},\ldots,\eta_{l},\nu_{1},\ldots,\nu_{i}$ are fixed), and direct extend q further, to a condition $q^{*}\geq^{*}q$, which has a system of extensions–

$$\langle q(\nu):\nu<g\rangle$$

as in the statement if the lemma.

It follows that, for every $\nu<g$,

$$q^{*}\Vdash\text{ if } \underset{\sim}{\mu}_{\alpha_{i+1}}(\xi)=\nu \text{ then } q(\nu)\in G\setminus\nu_{i} \text{ and } q(\nu)\setminus\nu\in e(\xi,\eta_{1},\ldots,\eta_{l},\nu_{1},\ldots,\nu_{i},\nu)$$

Therefore $\langle q(\nu)\colon\nu<g\rangle$ witnesses the fact that $q^{*}\in e(\xi,\eta_{1},\ldots,\eta_{l},\nu_{1},\ldots,\nu_{k})$.

$\square$ of lemma 4.3.13.

We now proceed towards the proof of theorem 4.3.3. We use the same notations as in the formulation of the theorem.

By induction, the following holds: For every $\xi,\eta_{1},\ldots,\eta_{l}$, the set $e(\xi,\eta_{1},\ldots,\eta_{l})\subseteq P\setminus\xi$ is $\leq^{*}$ dense open above conditions $q\in P\setminus\xi$ which force that–

$$\langle\theta_{f_{\beta_{1}}(\xi)},\ldots,\theta_{f_{\beta_{l}}(\xi)}\rangle=\langle\eta_{1},\ldots,\eta_{l}\rangle$$

and that–

$$F_{1}(\xi,\eta_{1},\ldots,\eta_{l})=\underset{\sim}{W}_{g_{1}(\xi,\eta_{1},\ldots,\eta_{l})}\text{ and } t_{1}(\xi,\eta_{1},\ldots,\eta_{l})=t^{q}_{g_{1}(\xi,\eta_{1},\ldots,\eta_{l})}$$

We would like to perform another step, and move from conditions in $P\setminus\xi$ to conditions in P. This might require extending the sequence generators $\beta_{1},\ldots,\beta_{l}$. We do this in the following lemma, which concludes the proof of theorem 4.3.3.

Lemma 4.3.15. *There exists $s<\omega$, a sequence of generators $\langle\beta'_1,\ldots,\beta'_s\rangle$ of i which extends $\langle\beta_1,\ldots,\beta_l\rangle$, and a system of conditions–*

$$\langle p\left(\xi,\eta'_1,\ldots,\eta'_s,\nu_1,\ldots,\nu_k\right):\eta'_1,\ldots,\eta'_s<\kappa,\quad \xi<\nu_1<\ldots<\nu_k\rangle$$

(all of them extend the condition $p\in G$ given in the statement of theorem 4.3.3), such that,

$$\begin{aligned}\{\xi<\kappa\colon & p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\right)\restriction_{\mu_{\alpha_k}(\xi)}\Vdash\\ & p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\right)\setminus\mu_{\alpha_k}(\xi)\in\\ & e\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\right)\ \textit{and-}\\ & p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\right)\in G\}\end{aligned}$$

Proof. Recall that $W=U_H$ is generated from the elementary embedding $i\colon V\to N$. Let us consider the set–

$$i\left(\langle\xi,\eta_1,\ldots,\eta_l\rangle\mapsto e\left(\xi,\eta_1,\ldots,\eta_l\right)\right)\left(\kappa,\beta_1,\ldots,\beta_l\right)\subseteq i(P)\setminus\kappa$$

it is $\leq^*$-dense open in $i(P)\setminus\kappa$, and thus meets a condition $r\in H$. Since $r\in N$, it can be represented using a sequence of generators $\langle\beta'_1,\ldots,\beta'_s\rangle$, on which we can assume that it contains $\langle\beta_1,\ldots,\beta_l\rangle$. Let–

$$\langle\xi,\eta'_1,\ldots,\eta'_s\rangle\mapsto r\left(\xi,\eta'_1,\ldots,\eta'_s\right)\in P\setminus\xi$$

be a function in V, such that–

$$r=i\left(\langle\xi,\eta'_1,\ldots,\eta'_s\rangle\mapsto r\left(\xi,\eta'_1,\ldots,\eta'_s\right)\right)\left(\kappa,\beta'_1,\ldots,\beta'_s\right)$$

Now, for every $\langle\xi,\eta'_1,\ldots,\eta'_s,\nu_1,\ldots,\nu_k\rangle$, let us define the condition $p\left(\xi,\eta'_1,\ldots,\eta'_s,\nu_1,\ldots,\nu_k\right)\in P$. We do this recursively, and define, for every $1\leq i\leq k$, a condition $p\left(\xi,\eta'_1,\ldots,\eta'_s,\nu_1,\ldots,\nu_i\right)\in P$. Simultaneously, we prove that–

$$\begin{aligned}\{\xi<\kappa\colon & p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi)\right)\restriction_{\mu_{\alpha_i}(\xi)}\Vdash\\ & p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi)\right)\setminus\mu_{\alpha_i}(\xi)\in\\ & e\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi)\right)\ \text{and-}\\ & p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi)\right)\in G\}\end{aligned}$$

This will complete the proof of the lemma, and thus, the proof of theorem 4.3.3.

- First, fix $\xi, \eta_1, \dots, \eta_s$, and let us define $p(\xi, \eta_1, \dots, \eta_s)$. If $p \restriction_\xi \Vdash r(\xi, \eta_1, \dots, \eta_s) \in e(\xi, \eta_1, \dots, \eta_l)$, set $p(\xi, \eta_1, \dots, \eta_s) = p \restriction_\xi {}^\frown r(\xi, \eta_1, \dots, \eta_s)$. Else, let $p(\xi, \eta_1, \dots, \eta_s)$ be an arbitrary condition above p. We argue that–

$$\{\xi < \kappa \colon p \restriction_\xi \Vdash r\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right) \in e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right) \text{ and } p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right) \in G\} \in W$$

  Recall that $r \in H$ was defined such that–

$$p \Vdash r \in i\left(\langle \xi, \eta_1, \dots, \eta_l \rangle \mapsto e(\xi, \eta_1, \dots, \eta_l)\right)(\kappa, \beta_1, \dots, \beta_l)$$

  applying the embedding $k \colon N \to M$ and reflecting down modulo W gives–

$$\{\xi < \kappa \colon p \restriction_\xi \Vdash r\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right) \in e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right)\} \in W$$

  Finally, $p \Vdash r \in H$ and thus $p \Vdash k(r) \in j_W(G)$, by lemma 4.2.17. Reflecting this down gives–

$$\{\xi < \kappa \colon p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \dots, \theta_{f_{\beta'_s}(\xi)}\right) \in G\} \in W$$

- Fix $\xi, \eta'_1, \dots, \eta'_s, \nu_1$ and let us define $p(\xi, \eta'_1, \dots, \eta'_s, \nu_1)$. Denote $g_1 = g_1(\xi, \eta'_1, \dots, \eta'_s)$.

  If $p(\xi, \eta'_1, \dots, \eta'_s) \restriction_\xi \Vdash p(\xi, \eta'_1, \dots, \eta'_s) \setminus \xi \in e(\xi, \eta'_1, \dots, \eta'_s)$, then $p(\xi, \eta'_1, \dots, \eta'_s) \restriction_\xi = p \restriction_\xi$ decides the statements–

$$F_1(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_j) = \underset{\sim}{W}_{g_1} \cap V \ , \ t^q_{g_1} = t_1(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)$$

  and, if it decides them positively, it forces that there exists a sequence $\langle q(\nu) \colon \nu < g_1 \rangle$ witnessing this. Define–

$$p(\xi, \eta'_1, \dots, \eta'_s, \nu_1) = p(\xi, \eta'_1, \dots, \eta'_s) \restriction_{\nu_1} {}^\frown q(\nu_1)$$

  If $p(\xi, \eta'_1, \dots, \eta'_s) \restriction_\xi \nVdash p(\xi, \eta'_1, \dots, \eta'_s) \setminus \xi \in e(\xi, \eta'_1, \dots, \eta'_s)$, or $p(\xi, \eta'_1, \dots, \eta'_s) \restriction_\xi \Vdash p(\xi, \eta'_1, \dots, \eta'_s) \setminus \xi \in e(\xi, \eta'_1, \dots, \eta'_s)$ but the statements–

$$F_1(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_j) = \underset{\sim}{W}_{g_1} \cap V \ , \ t^q_{g_1} = t_1(\xi, \eta_1, \dots, \eta_l, \nu_1, \dots, \nu_i)$$

  are decided negatively, let $p(\xi, \eta'_1, \dots, \eta'_s, \nu_1)$ be an arbitrary condition above $p(\xi, \eta'_1, \dots, \eta'_s)$.

We argue that–

$$\{\xi<\kappa\colon p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi)\right)\restriction_{\mu_{\alpha_1}(\xi)}\Vdash p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi)\right)\setminus\mu_{\alpha_1}(\xi)\in e\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi)\right)\text{ and- } p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi)\right)\in G\}$$

First, by the previous point,

$$\{\xi<\kappa\colon p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)}\right)\restriction_{\xi}\Vdash p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)}\right)\setminus\xi\in e\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)}\right)\}\in W$$

By the properties of the set $e\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)}\right)$, the condition–

$$p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)}\right)\restriction_{\xi}$$

decides the statements–

$$F_1\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)}\right)=\underset{\sim}{W}_{g_1}\cap V$$

and–

$$t_{g_1}^{p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)}\right)}=t_1\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)}\right)$$

Claim 4.3.16. *For a set of ξ-s in W, the above statements are decided in a positive way. Before the proof of the claim, let us proceed with our argument. By the claim and definition 4.3.4,*

$$p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi)\right)=p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)}\right)\restriction_{\mu_{\alpha_1}(\xi)}{}^\frown q\left(\mu_{\alpha_1}(\xi)\right)$$

and, by the properties of the set $e\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)}\right)$, the condition–

$$p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)}\right)$$

forces that–

$$p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi)\right)=p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)}\right)\restriction_{\mu_{\alpha_1}(\xi)}{}^\frown q\left(\mu_{\alpha_1}(\xi)\right)\in\underset{\sim}{G}$$

and–

$$p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\dots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi)\right)\setminus\mu_{\alpha_1}(\xi)=q\left(\mu_{\alpha_1}(\xi)\right)\in e\left(\xi,\theta_{f_{\beta'_1}(\xi)},\dots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi)\right)$$

Thus, for a set of ξ-s in W,

$$\{\xi<\kappa\colon p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\dots,\theta_{f_{\beta'_s}(\xi)}\right)\restriction_\xi\Vdash p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\dots,\theta_{f_{\beta'_s}(\xi)}\right)\setminus\xi\in e\left(\xi,\theta_{f_{\beta'_1}(\xi)},\dots,\theta_{f_{\beta'_s}(\xi)}\right)\}\in W$$

Which finishes the second step. Thus, it remains to prove claim 4.3.16:

Proof. Let us prove first that–

$$\{\xi<\kappa\colon p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\dots,\theta_{f_{\beta'_s}(\xi)}\right)\restriction_\xi\Vdash F_1\left(\xi,\theta_{f_{\beta'_1}(\xi)},\dots,\theta_{f_{\beta'_s}(\xi)}\right)=\underset{\sim}{W}_{g_1}\cap V\}$$

Assume otherwise. Then in $M\left[j_W(G)\right]$,

$$j_W\left(\langle\xi,\eta_1,\dots,\eta_s\rangle\mapsto F_1\left(\langle\xi,\eta_1,\dots,\eta_s\rangle\right)\right)\left(\kappa,j_{0,\alpha}\left(\beta'_1\right),\dots,j_{0,\alpha}\left(\beta'_s\right)\right)\neq\left[\xi\mapsto W_{g_1\left(\xi,\theta_{f_{\beta'_1}(\xi)},\dots,\theta_{f_{\beta'_s}(\xi)}\right)}\cap V\right]_W$$

but both sides are equal to $k_1\left(U_{\mu_{\alpha_1}}\right)$, contradicting property (D) of the embedding k_{α_1}.

Now let us prove that–

$$\{\xi<\kappa\colon p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\dots,\theta_{f_{\beta'_s}(\xi)}\right)\restriction_\xi\Vdash t_{g_1}^{p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\dots,\theta_{f_{\beta'_s}(\xi)}\right)}=t_1\left(\xi,\theta_{f_{\beta'_1}(\xi)},\dots,\theta_{f_{\beta'_s}(\xi)}\right)\}$$

Assume otherwise. Then the condition $s=j_W\left(\xi\mapsto p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\dots,\theta_{f_{\beta'_s}(\xi)}\right)\right)(\kappa)$ forces that–

$$t^s_{k_{\alpha_1}(\mu_{\alpha_1})}\neq k_{\alpha_1}\left(t_{\alpha_1}\right)=t_{\alpha_1}$$

Note that $s\in j_W(G)\restriction_{k_{\alpha_1}(\mu_{\alpha_1})}$ and t_{α_1} is the initial segment of the Prikry sequence of $k_{\alpha_1}\left(\mu_{\alpha_1}\right)$ below μ_{α_1} in $M\left[j_W(G)\right]$. Thus, one of the sequences $t^s_{k_{\alpha_1}(\mu_{\alpha_1})}$ and t_{α_1} is a strict initial segment of the other. By the second requirement in definition 4.3.4 , $s\restriction_{k_{\alpha_1}(\mu_{\alpha_1})}$ decides which one is an initial segment of the other. Now this yields a contradiction:

1. If t_{α_1} is a strict initial segment of $t^s_{k_{\alpha_1}(\mu_{\alpha_1})}$: Recall that $s = k_{\alpha_1}(s')$, where–

$$s' = j_{\alpha_1}\left(\langle \xi, \eta_1, \ldots, \eta_s\rangle \mapsto p\left(\xi, \eta_1, \ldots, \eta_s\right)\right)\left(\kappa, j_{0,\alpha_1}(\beta'_1), \ldots, j_{0,\alpha_1}\left(\beta'_s\right)\right)$$

   Then $s' \restriction_{\mu_{\alpha_1}}$ forces that t_{α_1} is a strict initial segment of $t^{s'}_{\mu_{\alpha_1}}$. Work over M_{α_1}. Let $\gamma < \mu_{\alpha_1}$ be an ordinal, forced by $s' \restriction_{\mu_{\alpha_1}}$ to be a bound on the first ordinal in $t^{s'}_{\mu_{\alpha_1}} \setminus t_{\alpha_1}$ (such a bound exists since the forcing $j_{\alpha_1}(P) \restriction_{\mu_{\alpha_1}}$ is μ_{α_1}-c.c. in M_{α_1}). Applying $k_{\alpha_1} : M_{\alpha_1} \to M$, $\gamma < \mu_{\alpha_1}$ is an upper bound on the first ordinal in $t^s_{k_{\alpha_1}(\mu_{\alpha_1})} \setminus t_{\alpha_1}$. However, in $M\left[j_W(G)\right]$, this element is μ_{α_1} itself, which is strictly above γ. A contradiction.

2. Else, $t^s_{k_{\alpha_1}(\mu_{\alpha_1})}$ is a strict initial segment of t_{α_1}: Denote $\gamma = \max\left(t_{\alpha_1}\right)$. Then, by definition 4.3.4, s forces that the initial segment of the Prikry sequence of $k_{\alpha_1}\left(\mu_{\alpha_1}\right)$ is $t^s_{k_{\alpha_1}(\mu_{\alpha_1})}$, followed by an element strictly above γ; in particular, t_{α_1} is not an initial segment of the Prikry sequence of $k_{\alpha_1}\left(\mu_{\alpha_1}\right)$ in $M\left[j_W(G)\right]$, which is a contradiction.

$\square$ of claim 4.3.16.

- Assume now that $1 \leq i < k$ is arbitrary, and for every $\xi, \eta'_1, \ldots, \eta'_s, \nu_1, \ldots, \nu_i$, a condition $p\left(\xi, \eta'_1, \ldots, \eta'_s, \nu_1, \ldots, \nu_i\right)$ is defined. Denote $g_{i+1} = g_{i+1}\left(\xi, \eta'_1, \ldots, \eta'_s, \nu_1, \ldots, \nu_i\right)$. For every $\nu_{i+1} < g_{i+1}$, let us define the condition $p\left(\xi, \eta'_1, \ldots, \eta'_s, \nu_1, \ldots, \nu_i, \nu_{i+1}\right)$. If $p\left(\xi, \eta'_1, \ldots, \eta'_s, \nu_1, \ldots, \nu_i\right) \restriction_{\nu_i} \Vdash$ $p\left(\xi, \eta'_1, \ldots, \eta'_s, \nu_1, \ldots, \nu_i\right) \setminus \nu_i \in e\left(\xi, \eta'_1, \ldots, \eta'_s, \nu_1, \ldots, \nu_i\right)$ and $p\left(\xi, \eta'_1, \ldots, \eta'_s, \nu_1, \ldots, \nu_i\right) \restriction_{\nu_i}$ forces the statements–

$$F_{i+1}\left(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i\right) = \underset{\sim}{W}_{g_{i+1}} \cap V \ , \ t^q_{g_{i+1}} = t_{i+1}\left(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_i\right)$$

  then $p\left(\xi, \eta'_1, \ldots, \eta'_s, \nu_1, \ldots, \nu_i\right) \restriction_{\nu_i}$ forces that there exists a sequence $\langle q(\nu) \colon \nu < g_{i+1}\rangle$ witnessing this. In this case, define–

$$p\left(\xi, \eta'_1, \ldots, \eta'_s, \nu_1, \ldots, \nu_i, \nu_{i+1}\right) = p\left(\xi, \eta'_1, \ldots, \eta'_s, \nu_1, \ldots, \nu_i\right) \restriction_{\nu_{i+1}} {}^\frown q(\nu_{i+1})$$

  Else, let $p\left(\xi, \eta'_1, \ldots, \eta'_s, \nu_1, \ldots, \nu_i, \nu_{i+1}\right)$ be an arbitrary condition which extends the condition $p\left(\xi, \eta'_1, \ldots, \eta'_s, \nu_1, \ldots, \nu_i\right)$.

Let us argue now that–

$$\begin{aligned}\{\xi<\kappa\colon\, & p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi),\mu_{\alpha_{i+1}}(\xi)\right)\restriction_{\mu_{\alpha_{i+1}}(\xi)}\Vdash \\ & p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi),\mu_{\alpha_{i+1}}(\xi)\right)\setminus\mu_{\alpha_{i+1}}(\xi)\in \\ & e\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi),\mu_{\alpha_{i+1}}(\xi)\right)\text{ and-} \\ & p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi),\mu_{\alpha_{i+1}}(\xi)\right)\in G\}\end{aligned}$$

We do this as in the previous point. First,

$$\begin{aligned}\{\xi<\kappa\colon\, & p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi)\right)\restriction_{\mu_{\alpha_i}(\xi)}\Vdash \\ & p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi)\right)\setminus\mu_{\alpha_i}(\xi)\in \\ & e\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi)\right)\}\in W\end{aligned}$$

Thus, for a set of ξ-s in W, the condition–

$$p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi)\right)\restriction_{\mu_{\alpha_i}(\xi)}$$

decides the statements–

$$\begin{aligned}& F_{i+1}\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi)\right)= \\ & \underset{\sim}{W}_{g_{i+1}\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi)\right)}\cap V\end{aligned}$$

and–

$$t^{p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi)\right)}_{g_{i+1}\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi)\right)}=t_{i+1}\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi)\right)$$

arguing as in claim 4.3.16, both statements are decided positively for a set of ξ-s in W. Thus,

$$\begin{aligned}& p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi),\mu_{\alpha_{i+1}}(\xi)\right)= \\ & p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi)\right)\restriction_{\mu_{\alpha_{i+1}}(\xi)}{}^\frown q\left(\mu_{\alpha_{i+1}}(\xi)\right)\end{aligned}$$

and the condition $q\left(\mu_{\alpha_{i+1}}(\xi)\right)$ is forced, by–

$$p\left(\xi,\theta_{f_{\beta_1'}(\xi)},\ldots,\theta_{f_{\beta_s'}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_i}(\xi)\right)$$

to be in–

$$G \setminus \mu_{\alpha_{i+1}}(\xi) \cap e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_i}(\xi), \mu_{\alpha_{i+1}}(\xi)\right)$$

Therefore,

$$\begin{aligned}\{\xi < \kappa \colon p&\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_i}(\xi), \mu_{\alpha_{i+1}}(\xi)\right) \restriction_{\mu_{\alpha_{i+1}}(\xi)} \Vdash \\ &p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_i}(\xi), \mu_{\alpha_{i+1}}(\xi)\right) \setminus \mu_{\alpha_{i+1}}(\xi) \in \\ &e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_i}(\xi), \mu_{\alpha_{i+1}}(\xi)\right) \text{ and-} \\ &p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_i}(\xi), \mu_{\alpha_{i+1}}(\xi)\right) \in G\}\end{aligned}$$

as desired.

$\square$ of lemma 4.3.15. $\square$ of theorem 4.3.3.

4.3.4 Properties of k_α

In this subsection we complete the proof of properties $(A) - (D)$ of k_α. After that, we will prove in lemma 4.3.20 that $k_{\kappa^*} \colon M_{\kappa^*} \to M$ is the identity, and conclude the proof of theorems 4.3.2 and 4.3.1.

Lemma 4.3.17. *$\mu_\alpha = crit(k_\alpha)$ is measurable in M_α. Moreover, μ_α is the least measurable above $sup\{\mu_\beta \colon \beta < \alpha\}$ which has cofinality above κ in V.*

Proof. Write $\mu = [f]_W$ and $\mu = j_\alpha(h)(\kappa, j_{0,\alpha}(\beta_1), \ldots, j_{0,\alpha}(\beta_k), \mu_{\alpha_1}, \ldots, \mu_{\alpha_m})$, for some $f \in V[G]$, $h \in V$, $\beta_1, \ldots, \beta_l$ generators of i and $\alpha_1 < \ldots < \alpha_k < \alpha$.

Since $\mu < k_\alpha(\mu)$, we can assume that for every $\xi < \kappa$,

$$f(\xi) < h\left(\xi, \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right)$$

and let $p \in G$ be a condition which forces this. Given $\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_k$, consider the set–

$$\begin{aligned} e(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_k) = \{r \in P \setminus \nu_k \colon &\text{ for some bounded subset } A \subseteq h(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_k), \\ &r \Vdash \underset{\sim}{f}(\xi) \in A\}\end{aligned}$$

Then $e(\xi, \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_k)$ is $\leq^*$-dense open above conditions which extend p and force that–

$$\langle \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi) \rangle = \langle \eta_1, \ldots, \eta_l, \nu_1, \ldots, \nu_k \rangle$$

By Theorem 4.3.3, the sequence $\langle \beta_1, \ldots, \beta_l \rangle$ can be extended to a sequence $\langle \beta'_1, \ldots, \beta'_s \rangle$, and p can be extended to a system of conditions,

$$\langle p\left(\xi, \eta_1, \ldots, \eta_s, \nu_1, \ldots, \nu_k\right) : \xi, \eta_1, \ldots, \eta_s < \kappa, \nu_1 < \ldots < \nu_k < \kappa \rangle$$

such that, for a set of ξ-s in W,

$$\begin{aligned} & p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \restriction_{\mu_{\alpha_k}(\xi)} \Vdash \\ & p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \setminus \mu_{\alpha_k}(\xi) \in \\ & e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \end{aligned}$$

and–

$$p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \in G$$

Assume now that $\langle \xi, \eta_1, \ldots, \eta_s, \nu_1, \ldots, \nu_k \rangle$ are given, such that–

$$\begin{aligned} p\left(\xi, \eta_1, \ldots, \eta_s, \nu_1, \ldots, \nu_k\right) \restriction_{\nu_k} \Vdash p\left(\xi, \eta_1, \ldots, \eta_s, \nu_1, \ldots, \nu_k\right) \setminus \nu_k \in \\ e\left(\xi, \eta_1, \ldots, \eta_s, \nu_1, \ldots, \nu_k\right) \end{aligned}$$

Let $\underset{\sim}{A}$ be a P_{ν_k}-name, forced by $p\left(\xi, \eta_1, \ldots, \eta_s, \nu_1, \ldots, \nu_k\right) \restriction_{\nu_k}$ to be a witness to the fact that $p\left(\xi, \vec{\eta}, \vec{\nu}\right) \setminus \nu_k \in e\left(\xi, \vec{\eta}, \vec{\nu}\right)$. Namely it is a bounded subset of $h\left(\xi, \vec{\eta}, \vec{\nu}\right)$, and $p\left(\xi, \vec{\eta}, \vec{\nu}\right) \setminus \nu_k \Vdash \underset{\sim}{f}(\xi) \in \underset{\sim}{A}$.

Let $A\left(\xi, \vec{\eta}, \vec{\nu}\right)$ be the set of ordinals $\gamma < h\left(\xi, \vec{\eta}, \vec{\nu}\right)$ such that, some $r \geq p\left(\xi, \vec{\eta}, \vec{\nu}\right) \restriction_{\nu_k}$ forces that $\gamma \in \underset{\sim}{A}$. Since $\nu_k < h\left(\xi, \vec{\eta}, \vec{\nu}\right)$, $A\left(\xi, \vec{\eta}, \vec{\nu}\right)$ is a bounded subset of $h\left(\xi, \vec{\eta}, \vec{\nu}\right)$. The function $\langle \xi, \vec{\eta}, \vec{\nu} \rangle \mapsto A\left(\xi, \vec{\eta}, \vec{\nu}\right)$ lies in V.

By the results of theorem 4.3.3, there exists a set of ξ-s in W for which–

$$\begin{aligned} G \ni & p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \Vdash \\ & \underset{\sim}{f}(\xi) \in A\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \end{aligned}$$

Thus, in $M\left[j_W(G)\right]$,

$$\begin{aligned} [f]_W \in & \left[\xi \mapsto A\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right)\right]_W = \\ & k_\alpha\left(j_\alpha\left(\langle \xi, \vec{\eta}, \vec{\nu} \rangle \mapsto A\left(\xi, \vec{\eta}, \vec{\nu}\right)\right)\left(\kappa, j_{0,\alpha}\left(\beta'_1\right), \ldots, j_{0,\alpha}\left(\beta'_s\right), \mu_{\alpha_1}, \ldots, \mu_{\alpha_k}\right)\right) \subseteq \operatorname{Im}\left(k_\alpha\right) \end{aligned}$$

where the last inclusion follows since–

$$j_\alpha\left(\langle \xi, \vec{\eta}, \vec{\nu} \rangle \mapsto A\left(\xi, \vec{\eta}, \vec{\nu}\right)\right)\left(\kappa, j_{0,\alpha}\left(\beta'_1\right), \ldots, j_{0,\alpha}\left(\beta'_s\right), \mu_{\alpha_1}, \ldots, \mu_{\alpha_k}\right)$$

is a bounded subset of–

$$\mu_\alpha = j_\alpha \left(\langle \xi, \vec{\eta}, \vec{\nu} \rangle \mapsto h\left(\xi, \vec{\eta}, \vec{\nu}\right) \right) \left(\kappa, j_{0,\alpha}\left(\beta'_1\right), \ldots, j_{0,\alpha}\left(\beta'_s\right), \mu_{\alpha_1}, \ldots, \mu_{\alpha_k}\right)$$

which is $\operatorname{crit}(k_\alpha)$.

Thus we proved that $\mu_\alpha \in \operatorname{Im}(k_\alpha)$, which is a contradiction. □

Lemma 4.3.18. *μ_α appears in the Prikry sequence added to $k_\alpha(\mu_\alpha)$ in $M[j_W(G)]$.*

Proof. In $M[H]$, denote by t^* the initial segment of the Prikry sequence of $k_\alpha(\mu_\alpha)$ which consists of all the ordinals below μ_α. Denote by n^* the length of t^*. Let $\langle \xi, \vec{\eta}, \vec{\nu} \rangle \mapsto t^*(\xi, \vec{\eta}, \vec{\nu})$ be a function in V such that–

$$t^* = j_\alpha \left(\langle \xi, \vec{\eta}, \vec{\nu} \rangle \mapsto t^*\left(\xi, \vec{\eta}, \vec{\nu}\right) \right) \left(\kappa, j_{0,\alpha}\left(\beta_1\right), \ldots, j_{0,\alpha}\left(\beta_l\right), \mu_{\alpha_0}, \ldots, \mu_{\alpha_k}\right)$$

(we assumed here that t^* can be represented using the same generators as μ_α. If this is not the case, modify the set of generators).

We can assume that for every $\langle \xi, \vec{\eta}, \vec{\nu} \rangle$, $t^*(\xi, \vec{\eta}, \vec{\nu})$ is a sequence of length n^*. Since $k_\alpha(t^*) = t^*$,

$$\left[\xi \mapsto t^*\left(\xi, \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right)\right]_W = t^*$$

In $V[G]$, denote, for every $\xi < \kappa$,

$$\mu_\alpha(\xi) = \text{the } \left(n^*+1\right)\text{-th element in the Prikry sequence of } h\left(\vec{\xi}, \theta_{f_{\beta_1}(\xi)}, \ldots, \theta_{f_{\beta_l}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right)$$

Clearly $[\xi \mapsto \mu_\alpha(\xi)]_W \geq \mu_\alpha$.

We argue that equality holds. We will prove that for every $\eta < [\xi \mapsto \mu_\alpha(\xi)]_W$, $\eta < \mu_\alpha$. Assume that such η is given, and let $f \in V[G]$ be a function such that $[f]_W = \eta$. Then we can assume that for every $\xi < \kappa$,

$$f(\xi) < \mu_\alpha(\xi)$$

and let $p \in G$ be a condition which forces this.

For every $\xi, \vec{\eta}, \vec{\nu}$, consider the set–

$$e\left(\xi, \vec{\eta}, \vec{\nu}\right) = \{ r \in P \setminus \nu_k \colon \exists \gamma < h\left(\xi, \vec{\eta}, \vec{\nu}\right),\ r \Vdash \text{ if } t^*\left(\xi, \vec{\eta}, \vec{\nu}\right) \text{ is an initial segment of the Prikry sequence of } h\left(\xi, \vec{\eta}, \vec{\nu}\right), \text{ then } \underset{\sim}{f}(\xi) < \gamma \}$$

then $e\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$ is $\leq^*$ dense open above conditions which force that–

$$\langle\theta_{f_{\beta_1}(\xi)},\ldots,\theta_{f_{\beta_l}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\rangle=\langle\eta_1,\ldots,\eta_l,\nu_1,\ldots,\nu_k\rangle$$

This, since, given a name for an element $\underset{\sim}{f}(\xi)$ which is forced to be strictly below $\mu_\alpha(\xi)$, (which is the element which appears right after $t^*(\xi,\vec{\eta},\vec{\nu})$ in the Prikry sequence of $h(\xi,\vec{\eta},\vec{\nu})$), the element can be decided by taking a direct extension.

By Theorem 4.3.3, the sequence $\langle\beta_1,\ldots,\beta_l\rangle$ can be extended to a sequence $\langle\beta'_1,\ldots,\beta'_s\rangle$, and p can be extended to a system of conditions,

$$\langle p(\xi,\eta_1,\ldots,\eta_s,\nu_1,\ldots,\nu_k):\xi,\eta_1,\ldots,\eta_s<\kappa,\nu_1<\ldots<\nu_k<\kappa\rangle$$

such that, for a set of ξ-s in W,

$$\begin{aligned}&p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\right)\restriction_{\mu_{\alpha_k}(\xi)}\Vdash\\&p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\right)\setminus\mu_{\alpha_k}(\xi)\in\\&e\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\right)\end{aligned}$$

and–

$$p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\right)\in G$$

Assume now that $\langle\xi,\vec{\eta},\vec{\nu}\rangle=\langle\xi,\eta_1,\ldots,\eta_s,\nu_1,\ldots,\nu_k\rangle$ are given, such that–

$$p(\xi,\vec{\eta},\vec{\nu})\restriction_{\nu_k}\Vdash p(\xi,\vec{\eta},\vec{\nu})\setminus\nu_k\in e(\xi,\vec{\eta},\vec{\nu})$$

Let $\underset{\sim}{\gamma}$ be a P_{ν_k}-name, forced by $p(\xi,\eta_1,\ldots,\eta_s,\nu_1,\ldots,\nu_k)\restriction_{\nu_k}$ to an ordinal below $h(\xi,\vec{\eta},\vec{\nu})$, such that $p(\xi,\vec{\eta},\vec{\nu})\setminus\nu_k\Vdash\underset{\sim}{f}(\xi)<\underset{\sim}{\gamma}$. Let $\gamma(\xi,\vec{\eta},\vec{\nu})$ be the supremum of the set of ordinals $\tau<h(\xi,\vec{\eta},\vec{\nu})$ such that, some $r\geq p(\xi,\vec{\eta},\vec{\nu})\restriction_{\nu_k}$ forces that $\underset{\sim}{\gamma}=\tau$. Since $\nu_k<h(\xi,\vec{\eta},\vec{\nu})$, $\gamma(\xi,\vec{\eta},\vec{\nu})<h(\xi,\vec{\eta},\vec{\nu})$. The function $\langle\xi,\vec{\eta},\vec{\nu}\rangle\mapsto\gamma(\xi,\vec{\eta},\vec{\nu})$ lies in V.

By the results of theorem 4.3.3, there exists a set of ξ-s in W for which–

$$\begin{aligned}G\ni p&\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\right)\Vdash\\&\text{if } t^*\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\right)\\&\text{is an initial segment of the Prikry sequence of}\\&h\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\right),\text{ then}\\&\underset{\sim}{f}(\xi)<\gamma\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\right)\end{aligned}$$

Thus, in $M[j_W(G)]$, where indeed t^* is an initial segment of the Prikry sequence of $k_\alpha(\mu_\alpha)$,

$$[f]_W \in \left[\xi \mapsto \gamma\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right)\right]_W =$$
$$k_\alpha\left(j_\alpha\left(\langle \xi, \vec{\eta}, \vec{\nu}\rangle \mapsto \gamma(\xi, \vec{\eta}, \vec{\nu})\right)(\kappa, j_{0,\alpha}(\beta'_1), \ldots, j_{0,\alpha}(\beta'_s), \mu_{\alpha_1}, \ldots, \mu_{\alpha_k})\right) < \mu_\alpha$$

as desired. □

Lemma 4.3.19. *Let $U_{\mu_\alpha} = \{X \subseteq \mu_\alpha \colon \mu_\alpha \in k_\alpha(X)\} \cap M_\alpha$. Then $U_{\mu_\alpha} \in M_\alpha$. Furthermore, $k_\alpha(U_{\mu_\alpha}) = j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha))$, where, for every $\delta \in \Delta$, $U_\delta = W_\delta \cap V$, for W_δ which is the measure used in the Prikry forcing at stage δ in the iteration P.*

Proof. We first prove that $j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha)) \in \mathrm{Im}(k_\alpha)$. Then, we will prove that the measure $F \in M_\alpha$ for which $j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha)) = k_\alpha(F)$ equals to U_{μ_α}.

In order to prove that $j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha)) \in \mathrm{Im}(k_\alpha)$, we prove that there exists a family $\mathcal{F} \in M_\alpha$ of measures on μ_α, with $|\mathcal{F}| < \mu_\alpha$, such that $j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha)) \in k_\alpha(F) = k''_\alpha \mathcal{F}$.

Fix, in V, an enumeration W of all the normal measures on measurable cardinals below κ. For every $\langle \xi, \vec{\eta}, \vec{\nu}\rangle$, let $\gamma(\xi, \vec{\eta}, \vec{\nu})$ be the index of $U_{h(\xi,\vec{\eta},\vec{\nu})}$ in this enumeration. Note that each measure $U_{h(\xi,\vec{\eta},\vec{\nu})}$ belongs to V, but the sequence $\langle U_{h(\xi,\vec{\eta},\vec{\nu})} \colon \xi, \vec{\eta}, \vec{\nu} < \kappa\rangle$ might be external to V. So the function $\langle \xi, \vec{\eta}, \vec{\nu}\rangle \mapsto \gamma(\xi, \vec{\eta}, \vec{\nu})$ doesn't necessarily belong to V.

Fix $\langle \xi, \vec{\eta}, \vec{\nu}\rangle$ and consider the set–

$$e(\xi, \vec{\eta}, \vec{\nu}) = \{r \in P \setminus \nu_k \colon \text{there exists a set of ordinals } A \text{ of cardinality strictly smaller than}$$
$$h(\xi, \vec{\eta}, \vec{\nu}), \text{ such that } r \restriction_{h(\xi,\vec{\eta},\vec{\nu})} \Vdash \gamma(\xi, \vec{\eta}, \vec{\nu}) \in A\}$$

Then $e(\xi, \vec{\eta}, \vec{\nu}) \subseteq P \setminus \nu_k$ is $\leq^*$-dense open, since $P \restriction_{h(\xi,\vec{\eta},\vec{\nu})}$ is $h(\xi, \vec{\eta}, \nu)$-c.c..

Now apply theorem 4.3.3 and argue as in the previous lemma: There exists (in V) a mapping $\langle \xi, \vec{\eta}, \vec{\nu}\rangle \mapsto A(\xi, \vec{\eta}, \vec{\nu})$ such that, in $M[j_W(G)]$,

$$\left[\xi \mapsto \gamma\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right)\right]_W \in$$
$$\left[\xi \mapsto A\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right)\right]_W =$$
$$k''_\alpha\left(j_\alpha\left(\langle \xi, \vec{\eta}, \vec{\nu}\rangle \mapsto A(\xi, \vec{\eta}, \vec{\nu})\right)(\kappa, j_{0,\alpha}(\beta'_1), \ldots, j_{0,\alpha}(\beta'_s), \mu_{\alpha_1}, \ldots, \mu_{\alpha_k})\right)$$

In M_α, let $\mathcal{F}$ be the set of measures on μ_α which are indexed in the enumeration $j_\alpha(W)$ by an index in the set $\mathcal{A} = j_\alpha(\langle \xi, \vec{\eta}, \vec{\nu}\rangle \mapsto A(\xi, \vec{\eta}, \vec{\nu}))(\kappa, j_{0,\alpha}(\beta'_1), \ldots, j_{0,\alpha}(\beta'_s), \mu_{\alpha_1}, \ldots, \mu_{\alpha_k})$. Note that

$|\mathcal{A}| < \mu_\alpha$ and thus $|\mathcal{F}| < \mu_\alpha$. Then $j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha))$ is enumerated by the ordinal–

$$\left[\xi \mapsto \gamma\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right)\right]_W \in k''_\alpha \mathcal{A}$$

and thus $j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha)) \in k''_\alpha \mathcal{F}$, as desired.

Let $F \in M_\alpha$ be a measure on μ_α such that–

$$j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha)) = k_\alpha(F)$$

Let us argue that $F = U_{\mu_\alpha}$. It suffices to prove that $F \subseteq U_{\mu_\alpha}$. Fix a set $X \in F$. Assume that–

$$X = j_\alpha(\langle \xi, \vec{\eta}, \vec{\nu} \rangle \mapsto X(\xi, \vec{\eta}, \vec{\nu}))(\kappa, j_{0,\alpha}(\beta_1), \ldots, j_{0,\alpha}(\beta_l), \mu_{\alpha_1}, \ldots, \mu_{\alpha_k})$$

(We assumed again that X can be represented using the same generators as μ_α. If this is not the case, modify the set of generators of μ_α). Then $k_\alpha(X) \in j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha))$.

As in the previous lemma, let n^* be the length of t^*, the initial segment of the Prikry sequence of $k_\alpha(\mu_\alpha)$ below μ_α. For every $\langle \xi, \vec{\eta}, \vec{\nu} \rangle$, let–

$$\begin{aligned} e(\xi, \vec{\eta}, \vec{\nu}) = \{r \in P \setminus \nu_k \colon\, & r \restriction_{h(\xi,\vec{\eta},\vec{\nu})} \parallel X(\xi, \vec{\eta}, \vec{\nu}) \in U_{h(\xi,\vec{\eta},\vec{\nu})}, \\ & \text{if it decides positively, then } r \restriction_{h(\xi,\vec{\eta},\vec{\nu})} \Vdash \underset{\sim}{A}^r_{h(\xi,\vec{\eta},\vec{\nu})} \subseteq \\ & X(\xi, \vec{\eta}, \vec{\nu});\ \text{else, } r \restriction_{h(\xi,\vec{\eta},\vec{\nu})} \Vdash \underset{\sim}{A}^r_{h(\xi,\vec{\eta},\vec{\nu})} \text{ is disjoint} \\ & \text{from } X(\xi, \vec{\eta}, \vec{\nu}).\ \text{Moreover, } r \restriction_{h(\xi,\vec{\eta},\vec{\nu})} \parallel \operatorname{lh}\left(t^r_{h(\xi,\vec{\eta},\vec{\nu})}\right) > n^*, \\ & \text{and if it decides positively, then there exists a bounded subset} \\ & A(\xi, \vec{\eta}, \vec{\nu}) \subseteq h(\xi, \vec{\eta}, \vec{\nu}) \text{ for which } r \restriction_{\xi,\vec{\eta},\vec{\nu}} \Vdash \text{ the } (n^*+1)\text{-th} \\ & \text{element of } t^r_{h(\xi,\vec{\eta},\vec{\nu})} \text{ belongs to } A(\xi, \vec{\eta}, \vec{\nu})\} \end{aligned}$$

By theorem 4.3.3, there exists a larger set of generators $\beta'_1, \ldots, \beta'_s$ and, for every $\langle \xi, \vec{\eta}, \vec{\nu} \rangle$, a condition $p(\langle \xi, \vec{\eta}, \vec{\nu} \rangle)$, such that, for a set of ξ-s in W,

$$\begin{aligned} & p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \restriction_{\mu_{\alpha_k}(\xi)} \Vdash \\ & p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \setminus \mu_{\alpha_k}(\xi) \in \\ & e\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \end{aligned}$$

and–

$$p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \in G$$

Let us argue first that for a set of ξ-s in W,

$$p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \restriction_{\mu_{\alpha_k}(\xi)}$$

decides that–

$$\mathrm{lh}\left(t^{p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\right)}_{h\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\right)}\right) \leq n^*$$

Indeed, assume otherwise. Let $A^*(\xi, \vec{\eta}, \vec{\nu})$ be the bounded subset of $h(\xi, \vec{\eta}, \vec{\nu})$ which consists of all the ordinals, which are forced by some extension of $p(\xi, \vec{\eta}, \vec{\nu}) \restriction_{\nu_k}$ to be in $A(\xi, \vec{\eta}, \vec{\nu})$ (whenever $p(\xi, \vec{\eta}, \vec{\nu})$ forces that the length of $t^{p(\xi,\vec{\eta},\vec{\nu})}_{h(\xi,\vec{\eta},\vec{\nu})}$ is greater than n^*). Then, in $M[j_W(G)]$,

$$\mu_\alpha \in k_\alpha\left(j_\alpha\left(\langle \xi, \vec{\eta}, \vec{\nu}\rangle \mapsto A^*(\xi, \vec{\eta}, \vec{\nu})\right)(\kappa, j_{0,\alpha}(\beta'_1), \ldots, j_{0,\alpha}(\beta'_s), \mu_{\alpha_1}, \ldots, \mu_{\alpha_k})\right)$$

But this is a contradiction, since $j_\alpha\left(\langle \xi, \vec{\eta}, \vec{\nu}\rangle \mapsto A^*(\xi, \vec{\eta}, \vec{\nu})\right)(\kappa, j_{0,\alpha}(\beta'_1), \ldots, j_{0,\alpha}(\beta'_s), \mu_{\alpha_1}, \ldots, \mu_{\alpha_k})$ is a bounded subset of μ_α.

Therefore, we can assume that–

$$p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \restriction_{\mu_{\alpha_k}(\xi)}$$

forces that–

$$\mathrm{lh}\left(t^{p\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\right)}_{h\left(\xi,\theta_{f_{\beta'_1}(\xi)},\ldots,\theta_{f_{\beta'_s}(\xi)},\mu_{\alpha_1}(\xi),\ldots,\mu_{\alpha_k}(\xi)\right)}\right) \leq n^*$$

Denote now $p^* = \left[\xi \mapsto p\left(\xi, \theta_{f_{\beta'_1}(\xi)}, \ldots, \theta_{f_{\beta'_s}(\xi)}, \mu_{\alpha_1}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right)\right]_W$. Then $p^* \restriction_{k_\alpha(\mu_\alpha)}$ forces that $\mu_\alpha \in \underset{\sim}{A}^{p^*}_{k_\alpha(\mu_\alpha)}$. By the definition of the sets $e(\xi, \vec{\eta}, \vec{\nu})$, the set $\underset{\sim}{A}^{p^*}_{k_\alpha(\mu_\alpha)}$ is forced to be either disjoint or contained in $k_\alpha(X)$. Since $k_\alpha(X) \in j_W(\delta \mapsto U_\delta)(k_\alpha(\mu_\alpha))$, it cannot be disjoint (again, by the definition of $e(\xi, \vec{\eta}, \vec{\nu})$). Therefore $\mu_\alpha \in k_\alpha(X)$ and thus $X \in U_{\mu_\alpha}$, as desired. $\square$.

Finally, let us argue that $j_{\kappa^*} = j_W \restriction_V$. Recall that $\kappa^* = i(\kappa)$, and note that $\kappa^* = \sup\{\mu_\alpha \colon \alpha < \kappa^*\}$.

Lemma 4.3.20. *$M = M_{\kappa^*}$, $j_W(\kappa) = i(\kappa)$ and $j_{\kappa^*} = j_W \restriction_V$.*

Remark 4.3.21. *In particular, if $i = j_U$ (namely W is simply generated) then $j_W(\kappa) = j_U(\kappa)$. On the other hand, possibly $j_U(\kappa) < i(\kappa)$, and then $j_W(\kappa) > j_U(\kappa)$.*

Proof. Define, similarly to $k_\alpha \colon M_\alpha \to M$, the embedding $k_{\kappa^*} \colon M_{\kappa^*} \to M$ as follows:

$$k_{\kappa^*}\left(j_{\kappa^*}(f)(\kappa, j_{0,\kappa^*}(\beta_1), \ldots, j_{0,\kappa^*}(\beta_l), \mu_{\alpha_1}, \ldots, \mu_{\alpha_k})\right) =$$
$$j_W(f)\left(\kappa, \theta_{[f_{\beta_1}(\xi)]_W}, \ldots, \theta_{[f_{\beta_l}(\xi)]_W}, \mu_{\alpha_1}, \ldots, \mu_{\alpha_m}\right)$$

for every $f \in V$, $\beta_1, \ldots, \beta_l$ generators of i and $\alpha_1 < \ldots < \alpha_m < \kappa^*$. Clearly $\operatorname{crit}(k_{\kappa^*}) \geq \kappa^*$. It suffices to prove that k_{κ^*} is the identity function.

Let τ be an ordinal, and let $f \in V[G]$ be a function such that $[f]_W = \tau$. By the κ-c.c. of P_κ, there exists $F \in V$ such that for every $\xi < \kappa$, $f(\xi) \in F(\xi)$ and $|F(\xi)| < \kappa$. Therefore, in $M[j_W(G)]$,

$$\tau = [f]_W \in [F]_W = k_{\kappa^*}(j_{\kappa^*}(F)(\kappa))$$

But–

$$|j_{\kappa^*}(F)(\kappa)| < j_{\kappa^*}(\kappa) = \kappa^* \leq \operatorname{crit}(k_{\kappa^*})$$

so $\eta \in \operatorname{Im}(k_{\kappa^*})$ as desired. □

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צמצומים של על-מסננים ושיכונים אלמנטריים מהרחבות גנריות למודל היסוד

עבודה זו

מוגשת כחלק מהדרישות

לקבלת תואר

״דוקטור לפילוסופיה״

על ידי

איל קפלן

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מרץ 2023