The Magidor Iteration and Restrictions of Ultrapowers to the Ground Model

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Abstract

We study the Magidor iteration of Prikry forcings below a measurable limit of measurables κ. We first characterize all the normal measures κ carries in the generic extension, building on and extending the main result of [1]. Then, for every such normal measure, we prove that the restriction of its ultrapower, from the generic extension to the ground model, is an iterated ultrapower of V by normal measures. This is done without core model theoretic assumptions; $\mathrm{GCH}_{\leq\kappa}$ in the ground model suffices.

Introduction

In this paper we revisit the Magidor iteration of Prikry forcings, which was first introduced by M. Magidor in his celebrated paper [10]. Its earliest application was to produce a model where the least strongly compact cardinal is the least measurable cardinal, settling a question of Tarski.

Assuming that κ is a measurable limit of measurables, the Magidor iteration can be used to destroy the measurability of every measurable cardinal $\alpha < \kappa$, while preserving cardinals and the measurability of κ itself. Given such an iteration P and a generic set $G \subseteq P$ over the ground model V, we consider the following questions:

1. What are the normal measures on κ in $V[G]$?
2. Given a normal measure $W \in V[G]$ on κ, let $j_W : V[G] \to M[H] \simeq \mathrm{Ult}(V[G], W)$ be its ultrapower embedding[1]. Is $j_W \restriction_V$ an iteration of V (by its measures or extenders)?
3. Given a normal measure $W \in V[G]$, is $j_W \restriction_V$ a definable class of V?

The first question is part of a larger body of work regarding the characterization and possible structure of the normal measures, carried by a cardinal κ, in forcing extensions which preserve its measurability. A very partial list of landmark results in this area include the works of Kunen-Paris [9], where the maximal possible number, κ^{++}, of normal measures on κ, is obtained; Friedman-Magidor [4], where it is proved that any intermediate value $1 < \lambda < \kappa^{++}$ can be obtained as the number of normal measures on κ; and Ben-Neria, [3], [2], where it is shown that every well-founded order can be realized as the Mitchell order on κ.

In the context of the Magidor iteration, this question was extensively studied by Ben-Neria in [1]. For every normal measure $U \in V$ on κ, he assigned a corresponding measure $U^\times \in V[G]$ on κ, and showed that the mapping $U \mapsto U^\times$ is a bijection between the set of normal measures

[1]We used the following well known fact: The model $\mathrm{Ult}(V[G], W)$ is isomorphic to a transitive model, which is, by elementarity, a generic extension of some ground model M with a generic set $H \subseteq j_W(G)$ over it.

on κ in V, and the set of normal measures on κ in $V[G]$. This was done under the assumptions that $0^{¶}$ does not exist and the ground model V is the core model. In this paper, we extend this result, weakening the assumption on the ground model V:

Theorem 0.1. *Assume $GCH_{\leq\kappa}$ holds in V. Let $W \in V[G]$ be a normal measure on κ. Then $W = U^{\times}$ for some normal measure $U \in V$ on κ. Moreover, the measures $\langle U^{\times} : U \in V$ is a normal measure on $\kappa\rangle$ are pairwise distinct.*

The proof relies on some of the methods presented by Ben-Neria in [1]; however, the core-model theoretic aspects of the argument are replaced with the tools developed in [6].

The second question is motivated by key results and ideas from inner model theory. Assume that the core mode $\mathcal{K}$ exists and $j : V \to N$ is an arbitrary elementary embedding, where N is transitive. Under limitations on the variety of large cardinals available in the universe, the restriction $j\restriction_{\mathcal{K}}$ is an iteration of $\mathcal{K}$ by its measures and extenders. For instance, if there is no inner model with a cardinal α of Mitchell order $o(\alpha) = \alpha^{++}$, then, by results of Mitchell [11], $j\restriction_{\mathcal{K}}$ is an iteration of $\mathcal{K}$ by its measures; Assuming that there is no inner model with a strong cardinal, $j\restriction_{\mathcal{K}}$ is an iteration of $\mathcal{K}$ by its extenders [8]. In our context, assuming that $V = \mathcal{K}$ is the core model and $G \subseteq P$ is generic for the Magidor iteration over it, the ultrapower embedding $j_W : V[G] \to M[H]$ restricts to an iteration of $V = \mathcal{K}$, provided that there is no inner model with a Woodin cardinal (see [12]).

The main question that rises is to what extent the structure of $j_W\restriction_V$, as an iteration of V, depends on properties of V rather than properties of the Magidor iteration itself. It turns out that the core model theoretic assumptions imposed on V can be entirely omitted:

Theorem 0.2. *Assume $GCH_{\leq\kappa}$ holds in V. Let $W \in V[G]$ be a normal measure on κ. Then $j_W\restriction_V$ is an iterated ultrapower of V by normal measures.*

Moreover, a concrete description of $j_W\restriction_V$ as an iterated ultrapower is given. This uses and extends ideas appearing in [7], where iterations of Prikry forcings were considered under the simpler nonstationary support.

The answer to the third question depends on the choice of the normal measures used along the iteration to singularize the measurables of V below κ. In general, $j_W\restriction_V$ may not be definable in V (see remark 3.25, and, more generally, section 5.2 in [6]). We provide a sufficient condition for definability of $j_W\restriction_V$ as a class of V. By Theorem 0.1, given a measurable $\alpha < \kappa$, the measure used in the Prikry forcing at stage α in the iteration P must have the form $U^{\times}_{\alpha} = (U_{\alpha})^{\times}$, for some normal measure U_{α} on α in V. Denote $\vec{\mathcal{U}} = \langle U_{\alpha} : \alpha < \kappa,\ \alpha \text{ is measurable in } V\rangle$. Then:

Theorem 0.3. *Assume $GCH_{\leq\kappa}$ holds in V. If $\vec{\mathcal{U}} \in V$ then $j_W\restriction_V$ is a definable class of V.*

We remark that it is not necessarily the case that $\vec{\mathcal{U}} \in V$, even if $j_W\restriction_V$ is a definable class of V (see remark 3.25).

This paper is organized as follows: In the first section we present the forcing and its basic properties. In section 2 we prove theorem 0.1. In section 3 we prove theorems 0.2 and 0.3, and completely describe the Prikry sequences added to measurables of M above κ in H.

Conventions and Notations. We assume throughout this paper that $\mathrm{GCH}_{\leq\kappa}$ holds in V. In forcing, we use the following convention: Given pair of conditions p, q in a forcing notion, $p \geq q$ means that p extends q, namely provides more information than q. Finally, given a measure $U \in V$ on some measurable, we denote by M_U the transitive collapse of the ultrapower $\mathrm{Ult}(V, U)$.

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1 The Forcing

Definition 1.1. *An iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \colon \alpha \le \kappa\ ,\ \beta < \kappa \rangle$ is called a full support (Magidor) iteration of Prikry-type forcings if and only if, for every $\alpha \le \kappa$ and $p \in P_\alpha$,*

1. *p is a function with domain α such that for every $\beta < \alpha$, $p \restriction \beta \in P_\beta$, $p \restriction \beta \Vdash p(\beta) \in \underset{\sim}{Q}_\beta$ and $\langle \underset{\sim}{Q}_\beta, \underset{\sim}{\le}_{\underset{\sim}{Q}_\beta}, \underset{\sim}{\le}^*_{\underset{\sim}{Q}_\beta} \rangle$ is a Prikry-type forcing.*
2. *There exists a finite set $b \subseteq \kappa$ such that for every $\beta \notin b$, $p \restriction_\beta \Vdash p(\beta) \underset{\sim}{\ge}^*_\beta \underset{\sim}{0}_{Q_\beta}$, where $\underset{\sim}{\ge}^*_\beta$ is the direct extension order of $\underset{\sim}{Q}_\beta$.*

Suppose that $p, q \in P_\alpha$. Then $p \ge q$, which means that p extends q, holds if and only if:

1. *For every $\beta \le \alpha$, $p \restriction \beta \Vdash p(\beta) \ge_\beta q(\beta)$ (where $\ge_\beta$ is the order of Q_β).*
2. *There is a finite subset $b \subseteq \alpha$, such that for every $\beta \in \alpha \setminus b$, $p \restriction \beta \Vdash p(\beta) \ge^*_\beta q(\beta)$ (where $\ge^*_\beta$ is the direct extension order of Q_β).*

If $b = \emptyset$, we say that p is a direct extension of q, and denote it by $p \ge^* q$.

Let $\langle P_\alpha, \underset{\sim}{Q}_\beta \colon \alpha \le \kappa\ ,\ \beta < \kappa \rangle$ be a full support iteration of Prikry forcings, such that, for every V-measurable cardinal, α, $\underset{\sim}{Q}_\alpha$ is non-trivial, and is forced to be Prikry forcing with a given P_α-name for a normal measure on α (we will prove in lemma 2.2 that α remains measurable after forcing with P_α). If α is not measurable in V, Q_α is the trivial forcing.

We denote $\Delta = \{\alpha < \kappa \colon \alpha \text{ is measurable in } V\}$. For every $\alpha \in \Delta$, let $\underset{\sim}{W}_\alpha$ be the P_α-name for a normal measure on α, which is forced by P_α to be the measure used in the Prikry forcing $\underset{\sim}{Q}_\alpha$. Assume that $p \in P_\kappa$ is a given condition and $\alpha \in \Delta$. We denote by $\underset{\sim}{t}^p_\alpha$ and $\underset{\sim}{A}^p_\alpha$ the P_α-names such that $p \restriction_\alpha \Vdash p(\alpha) = \langle \underset{\sim}{t}^p_\alpha, \underset{\sim}{A}^p_\alpha \rangle$. In $V[G]$, we denote by $d \colon \Delta \to \kappa$ the function which maps each former measurable in Δ to the first element in its Prikry sequence. Finally, we adopt the following useful notation, introduced by O. Ben-Neria in [1]: Given a condition $p \in P_\kappa$ and $\alpha < \kappa$, let $p^{-\alpha} \ge^* p$ be the condition p^* which satisfies, for every measurable $\xi > \alpha$,

$$p^* \restriction_\xi \Vdash \underset{\sim}{A}^{p^*}_\xi = \underset{\sim}{A}^p_\xi \setminus (\alpha + 1)$$

The following lemma is standard (see [5] for example):

Lemma 1.2. *$P = P_\kappa$ satisfies the Prikry property.*

The main ideas in the proof of the Prikry property of P_κ appear also in the proof of the following Fusion property:

Lemma 1.3 (Fusion Lemma)**.** *Let $\delta \le \kappa$ be a limit ordinal and $p \in P_\delta$. For every $\alpha < \delta$, let $q(\alpha)$ be a P_α-name such that $p \restriction_\alpha \Vdash q(\alpha) \ge^* p \setminus \alpha$. Then there exist $p^* \ge^* p$ such that for every $\alpha < \delta$,*

$$p^* \restriction_\alpha \Vdash (p^* \setminus \alpha)^{-\alpha} \ge^* q(\alpha)$$

Before proving the lemma, let us state an immediate useful corollary of it.

Corollary 1.4. *Let $\delta \le \kappa$ be a limit ordinal and $p \in P_\delta$. For every $\alpha < \delta$, let $e(\alpha)$ be a P_α-name such that–*

$$p \restriction_\alpha \Vdash \text{"}e(\alpha) \textit{ is a dense open subset of } P \setminus \alpha \textit{ above } p \setminus \alpha,$$
$$\textit{with respect to the direct extension order."}$$

Then there exist $p^* \ge^* p$ such that for every $\alpha < \delta$,

$$p^* \restriction_\alpha \Vdash (p^* \setminus \alpha)^{-\alpha} \in e(\alpha)$$

Proof of lemma 1.3. Define a sequence $\langle p_\xi \colon \xi > \delta\rangle$ of direct extensions of p, such that for every $\xi < \delta$, $p_\xi \restriction_\xi \Vdash p_\xi \setminus \xi \geq^* q(\xi)$, and, for every $\eta < \xi < \delta$,

1. $p \leq^* p_\eta \leq^* (p_\xi)^{-\eta}$.
2. $p_\eta \restriction_\eta = p_\xi \restriction_\eta$.

Take $p_0 = p$. Assume that $\xi < \kappa$ and $\langle p_\eta \colon \eta < \xi\rangle$ have been defined. Let us define p_ξ. First, set–

$$p_\xi \restriction_\xi = \bigcup_{\xi' < \xi} p_{\xi'} \restriction_{\xi'}$$

We now define a P_ξ-name for a condition $r \in P \setminus \xi$. If ξ is non-measurable, $r(\xi)$ (the value of r at coordinate ξ) is trivial. If it is: Let $\underset{\sim}{t}$ be a P_ξ-name, and, for every $\xi' < \xi$, take P_ξ-names $\underset{\sim}{A}_{\xi'}$ such that $p_\xi \restriction_\xi \Vdash p_{\xi'}(\xi) = \langle \underset{\sim}{t}, \underset{\sim}{A}_{\xi'}\rangle$. Set $r(\xi) = \langle \underset{\sim}{t}, \triangle_{\xi'<\xi} \underset{\sim}{A}_{\xi'}\rangle$. Finally, let $r \setminus (\xi+1)$ be a direct extension of all the conditions $\langle p_{\xi'} \setminus (\xi+1) : \xi' < \xi\rangle$ (the direct extension order above ξ is more than ξ-closed). This defines r. Since every pair of direct extensions of $p \setminus \xi$ have a common direct extension, we can pick $p_\xi \setminus \xi$ such that it direct extends both r and $q(\xi)$. Note that $\underset{\sim}{A}_\xi^{p_\xi} \subseteq \triangle_{\xi'<\xi} \underset{\sim}{A}_\xi^{p_{\xi'}}$, and thus, for every $\eta < \xi$, $\underset{\sim}{A}_\xi^{p_\xi} \setminus (\eta+1) \subseteq \underset{\sim}{A}_\xi^{p_\eta}$. Thus $p_\eta \leq^* (p_\xi)^{-\eta}$.

This finishes the construction. Define $p^* = \bigcup_{\xi<\delta} p_\xi \restriction_\xi$. We claim that p^* is as desired. Let $\alpha < \delta$. Then $p^* \restriction_\alpha = p_\alpha \restriction_\alpha$. Thus, this condition forces that $(p_\alpha \setminus \alpha)^{-\alpha} \in e(\alpha)$. It also forces that $(p^* \setminus \alpha)^{-\alpha}$ direct extends $(p_\alpha \setminus \alpha)^{-\alpha}$, and thus it direct extends $q(\alpha)$, as desired. $\square$

Lemma 1.5. *$P = P_\kappa$ preserves cardinals.*

Proof. We prove by induction that for every $\delta \leq \kappa$, P_δ preserves cardinals. This is clear for successor values of δ. By $\mathrm{GCH}_{\leq\kappa}$, this is clear as well if δ is not a limit of measurables. Thus, let us assume that $\delta \leq \kappa$ is a limit of measurables and μ is a cardinal. If $\mu < \delta$, factor $P_\delta = P_{<\mu} * \underset{\sim}{Q}_\mu * P_{>\mu}$. Since the direct extension order of $P_{>\mu}$ is more than μ-closed, it preserves μ; $\underset{\sim}{Q}_\mu$ preserves μ because it is either trivial or a Prikry forcing; finally, by induction, $P_{<\mu}$ preserves μ. If $\mu = \delta$, then μ is a limit of measurables, each of them is preserved by induction. If $\mu > \delta^+$, μ is preserved since $|P_\delta| = \delta^+$ by $\mathrm{GCH}_{\leq\kappa}$. Thus, it suffices to prove that P_δ preserves δ^+ for every limit of measurables δ. It suffices to prove that P_δ has the $\delta^+ - c.c.$: For any antichain $A \subseteq P_\delta$ of cardinality δ^+, there exists a subset $A' \subseteq A$ of cardinality δ^+, such that the following holds: There exists a finite set $b \subseteq \delta$, and, for every $\alpha \in b$, a P_α-name for a finite increasing sequence $\underset{\sim}{t}_\alpha \in [\alpha]^{<\omega}$, such that–

$$\forall p \in A' \ \forall \beta \in \delta \setminus b \ , p \restriction_\beta \Vdash p(\beta) \geq^* 0_Q$$

and–

$$\forall p \in A' \ \forall \alpha \in b \ \exists \underset{\sim}{A}, p \restriction_\alpha \Vdash p(\alpha) = \langle \underset{\sim}{t}_\alpha, \underset{\sim}{A}\rangle$$

Given these properties, every pair of conditions in A' are compatible, which is a contradiction. $\square$

Lemma 1.6. *$P = P_\kappa$ doesn't add fresh subsets of κ, κ^+.*

The above lemma is proved, for example, in [6]. We remark that this proof uses the fact that some normal measure on κ in V extend to a normal measure in $V[G]$, and this is indeed the case (this is well known, and in any case, will be proved in the next section in lemma 2.2. The proof will not rely on the current lemma or its consequences).

In [6] it is proved that, if a forcing notion P preserves cardinals and does not add fresh subsets to cardinals in the interval $\left[\kappa, (2^\kappa)^V\right]$, then every κ-complete ultrafilter in the generic extension extends a κ-complete ultrafilter of V. Since we assume $\mathrm{GCH}_{\leq\kappa}$, the following follows:

Corollary 1.7. *Let $G \subseteq P_\kappa$ be generic over V, and let $W \in V[G]$ be a κ-complete ultrafilter on κ. Then $W \cap V \in V$.*

We conclude this section by proving a property of $P = P_\kappa$, which will be applied several times throughout this paper.

Lemma 1.8. *Let $\delta \leq \kappa$ be an inaccessible cardinal. Let $p \in P_\delta$ and assume that $\underset{\sim}{\alpha}$ is a P_δ-name for an ordinal. Then there exists $p^* \geq^* p$ and a set $A \in V$ with $|A| < \delta$ such that $p^* \Vdash \underset{\sim}{\alpha} \in A$.*

Proof. Denote by D the dense open subset of P_δ which consists of conditions which decide the value of $\underset{\sim}{\alpha}$. We will apply on D the following claim:

Claim 1.9. *Let $\delta \leq \kappa$ be a limit ordinal and let $D \subseteq P_\delta$ be a dense open subset of P_δ. Assume that $p \in P_\delta$. Then there exists $p^* \geq^* p$ such that for every $p^* \leq q \in D$,*

$$q \restriction_{\gamma+1}{}^\frown (p^* \setminus (\gamma+1))^{-(\gamma+1)} \in D$$

where γ is the maximal coordinate which satisfies–

$$q \restriction_\gamma \Vdash \text{"}q(\gamma) \text{ is not a direct extension of } p^*(\gamma)\text{"}$$

(and, if such γ does not exist, then $\gamma = 0$).

Proof. Fix a non-measurable $\alpha < \delta$ and $G_\alpha \subseteq P_\alpha$ generic over V such that $p \restriction_\alpha \in G_\alpha$. Given $p \restriction_\alpha \leq q \in G_\alpha$, we define a subset of $P \setminus \alpha$ which is $\leq^*$-dense open above $p \setminus \alpha$:

$$e_q(\alpha) = \{r \in P \setminus \alpha \colon q^\frown r \in D \text{ or } (\forall r' \geq^* r,\ q^\frown r' \notin D)\}$$

Since α is non-measurable, the direct extension order of $P \setminus \alpha$ is more than $|G_\alpha|^+$-distributive. Let $e(\alpha)$ be a P_α-name for the set–

$$e(\alpha) = \bigcap_{q \in G_\alpha} e_q(\alpha)$$

then $p \restriction_\alpha$ forces that $e(\alpha)$ is $\leq^*$-dense open above $p \setminus \alpha$.

Apply lemma 1.3. Let $p^* \geq^* p$ be such that, for every non-measurable $\alpha < \delta$,

$$p^* \restriction_\alpha \Vdash (p^* \setminus \alpha)^{-\alpha} \in e(\alpha)$$

Assume now that $p^* \leq q \in D$. Let γ be as in the formulation of the claim. Then $\gamma + 1$ is not measurable, so–

$$p^* \restriction_{\gamma+1} \Vdash (p^* \setminus (\gamma+1))^{-\gamma+1} \in e(\gamma+1)$$

In particular,

$$q \restriction_{\gamma+1} \Vdash (p^* \setminus (\gamma+1))^{-\gamma+1} \in e(\gamma+1)$$

Finally, since there exists a direct extension $r' = q \setminus (\gamma+1) \geq^* p^* \setminus (\gamma+1)$ such that $q \restriction_{\gamma+1}{}^\frown r' \in D$, it follows that $q \restriction_{\gamma+1}{}^\frown (p^* \setminus (\gamma+1))^{-\gamma+1} \in D$, as desired. $\square$

Pick a direct extension $q \geq^* p$, by applying the claim on the set D of conditions deciding the value of $\underset{\sim}{\alpha}$. We will construct below a direct extension $q^* \geq^* q$; After this is done, we will prove that q^* has a direct extension $p^* \geq^* q^*$ as desired in the lemma. Namely, p^* satisfies that for some set of ordinals A with $|A| < \delta$, $p^* \Vdash \underset{\sim}{\alpha} \in A$.

First, let us construct $q^* \geq^* q$. Assume that $\gamma < \delta$, and $q^* \restriction_\gamma$ has been defined. To define $q^*(\gamma)$, we shrink the set $\underset{\sim}{A}^q_\gamma$. We shrink it to a set $A \in W_\gamma$, such that, for every $n < \omega$, exactly one of the following holds: Either for every $s \in [A]^n$, there exists a set of ordinals A_s with $|A_s| < \delta$, such that–

$$\langle t^q_\gamma{}^\frown s, A \setminus \max(s)\rangle^\frown (q \setminus (\gamma+1))^{-\gamma+1} \Vdash \underset{\sim}{\alpha} \in A_s$$

or, there is no such s.

This results in a direct extension $q^* \geq^* q$. It suffices to prove that q^* has a direct extension p^* which belongs to D. Assume otherwise. Let $r \geq q^*$ be a condition in D, which is chosen with the least number of non-direct extensions. Let γ be the maximal coordinate in which a non-direct extension was taken in the extension $r \geq q^*$. Clearly $r \geq q$, and in this extension, as well, γ is the maximal coordinate in which a non-direct extension is taken. Thus, by the choice of q,

$$r \restriction_{\gamma+1}{}^\frown (q \setminus (\gamma+1))^{-\gamma+1} \in D$$

Let $n < \omega$ be such that $r \restriction_\gamma$ forces that $\mathrm{lh}\left(t^r_\gamma\right) = n + \mathrm{lh}\left(t^q_\gamma\right)$. Then $r \restriction_\gamma$ forces that for every $s \in \left[\underset{\sim}{A}^r_\gamma\right]^n$, there exists a set A_s with $|A_s| < \delta$, such that–

$$\langle t^q_\gamma{}^\frown s, \underset{\sim}{A}^r_\gamma \setminus \max(s)\rangle^\frown (q \setminus (\gamma+1))^{-\gamma+1} \Vdash \underset{\sim}{\alpha} \in A_s$$

By taking union on the possible values of the sets A_s as above, there exists a set $A \in V$ with $|A| < \delta$ such that–

$$r \restriction_\gamma{}^\frown \langle \underset{\sim}{t}^q_\gamma, \underset{\sim}{A}^r_\gamma\rangle^\frown (q \setminus (\gamma+1))^{-\gamma+1} \Vdash \underset{\sim}{\alpha} \in A$$

and this contradicts the minimality of the number of non-direct extensions in the choice of $r \geq q^*$. $\square$

Corollary 1.10. *Assume that $\delta \leq \kappa$ is inaccessible, $p \in P_\delta$ and let $\underset{\sim}{f}$ be a P_δ-name for a function from δ to the ordinals. Then there exists $p^* \geq^* p$ and a function $F \colon \delta \to [Ord]^{<\delta}$ in V, such that for every $\alpha < \delta$,*

$$(p^*)^{-\alpha} \Vdash \underset{\sim}{f}(\alpha) \in F(\alpha)$$

Proof. For every $\alpha < \delta$, set–

$$e(\alpha) = \{r \in P \setminus \alpha \colon \text{there exists } A \subseteq \mathrm{Ord} \text{ with } |A| < \delta \text{ such that } r \Vdash \underset{\sim}{f}(\alpha) \in A\}$$

by lemma 1.8, $e(\alpha)$ is $\leq^*$-dense open. Thus, by Fusion, there exists $p^* \geq^* p$ such that for every $\alpha < \delta$,

$$p^* \restriction_\alpha \Vdash \text{there exists } A_\alpha \subseteq \mathrm{Ord} \text{ with } |A_\alpha| < \delta \text{ such that } (p^* \setminus \alpha)^{-\alpha} \Vdash \underset{\sim}{f}(\alpha) \in A_\alpha$$

Finally, for every $\alpha < \delta$, let $F(\alpha) = \{\beta \colon \exists q \geq p^* \restriction_\alpha,\ r \Vdash \beta \in \underset{\sim}{A}_\alpha\}$. Then $(p^*)^{-\alpha} \Vdash \underset{\sim}{f}(\alpha) \in F(\alpha)$ and $|F(\alpha)| < \delta$, as desired. $\square$

2 Normal Measures in the Generic Extension

This section is devoted to the proof of theorem 0.1. The same result was first observed by O. Ben-Neria in [1], assuming that V is the core model and there is no inner mode with overlapping extenders. We will reduce the assumptions on V to $\mathrm{GCH}_{\leq\kappa}$.

Throughout this section, we will extensively use arguments and notations introduced in [1]: For every normal measure on κ, $U \in V$, we will define a measure $U^* \in V[G]$ which extends U. It will turn out that U^* is normal if and only if $o(U) = 0$. Let $U^\times$ be the normal measure below U^* in the Rudin-Keisler order. We will prove that every normal measure on κ in $V[G]$ has the form $U^\times$ for some $U \in V$.

We prove theorem 0.1 by induction. Thus, we assume in this section that for every $\xi < \kappa$, the measure W_ξ used to singularize ξ, already has the form $U_\xi^\times$ for some normal measure $U_\xi \in V$ on ξ.

Remark 2.1. *In [1], as in other applications of the Magidor iteration, it was assumed that the measures $\langle W_\xi : \xi \in \Delta \rangle$, which were used to singularize the measurables of Δ, are all derived from normal measures of Mitchell order 0 (in the sense that, for every $\xi \in \Delta$, there exists $U_\xi \in V$ of order 0, such that $W_\xi = U_\xi^*$). We do not assume this in the current paper. Each measure W_ξ has, by induction, the form $U_\xi^\times$ for some normal measure $U_\xi \in V$, but U_ξ does not necessarily has Mitchell order 0.*

We start by extending every normal measure $U \in V$ on κ, to a measure $U^* \in V[G]$. For every P_κ-name $\underset{\sim}{A}$ for a subset of κ, $(\underset{\sim}{A})_G \in U^*$ if and only if, for some $p \in G$,

$$\{\xi < \kappa \colon p^{-\xi} \Vdash \check{\xi} \in \underset{\sim}{A}\} \in U$$

or simply $(j_U(p))^{-\kappa} \Vdash \check{\kappa} \in j_U(\underset{\sim}{A})$ in M_U.

Lemma 2.2. *U^* is a measure on κ in $V[G]$ which extends U. Moreover, U^* is normal if and only if U has Mitchell order 0 in V.*

Proof. It's not hard to verify that U^* is a filter which extends U. Let us prove that it is a κ-complete ultrafilter. Assume that $\langle \underset{\sim}{A}_\xi : \xi < \delta \rangle$ is forced by a condition $p \in G$ to be a partition of κ, for some $\delta < \kappa$. Assume that q is an arbitrary condition above p. For every $\alpha \in (\delta, \kappa)$, consider the P_α-name for the following set $e(\alpha)$, which is forced by $q \restriction_\alpha$ to be $\leq^*$-dense open above $q \setminus \alpha$,

$$e(\alpha) = \{r \geq^* q \setminus \alpha \colon \exists \xi^* < \delta,\ r \Vdash \check{\alpha} \in \underset{\sim}{A}_{\xi^*}\}$$

by lemma 1.3, there exists $p^* \in G$ above p, such that for every $\alpha \in (\delta, \kappa)$,

$$p^* \restriction_\alpha \Vdash \exists \xi^* < \delta,\ (p^* \setminus \alpha)^{-\alpha} \Vdash \check{\alpha} \in \underset{\sim}{A}_\xi$$

and thus–

$$p^* \Vdash \exists \xi^* < \delta,\ (j_U(p^*))^{-\kappa} \setminus \kappa \Vdash \check{\kappa} \in j_U(\underset{\sim}{A}_{\xi^*})$$

by extending p^* to a stronger condition in G, we can assume that p^* decides the value of ξ^*, and so, for some $\xi^* < \kappa$,

$$(j_U(p^*))^{-\kappa} \Vdash \check{\kappa} \in j_U(\underset{\sim}{A}_{\xi^*})$$

as desired.

Let us assume that U has Mitchell order 0. Let $\underset{\sim}{f}$ be a P_κ-name for a regressive function, as forced by some $p \in G$. We use a similar argument as before, but now $e(\alpha)$ is defined for every non-measurable α, to be the name for the following set, which is forced by any extension of $p \restriction_\alpha$ to be $\leq^*$-dense open above $p \setminus \alpha$:

$$e(\alpha) = \{r \in P \setminus \alpha \colon \exists \xi^* < \alpha,\ r \Vdash \underset{\sim}{f}(\alpha) = \xi^*\}$$

where we used the fact that α is not measurable, and thus $\langle P \setminus \alpha, \leq^* \rangle$ is more than α-closed. Thus, there exists $p^* \in G$ such that-

$$p^* \Vdash \exists \xi^* < \kappa,\ (j_U(p^*) \setminus \kappa)^{-\kappa} \Vdash j_U(\underset{\sim}{f})(\kappa) = \xi^*$$

By extending p^* to a condition in G, we can assume that p^* decides the value of ξ^*. Thus, $\{\xi < \kappa \colon f(\xi) = \xi^*\} \in U^*$, as desired.

Finally, assume that U^* is normal. Let $j_{U^*} \colon V[G] \to M[H]$ be the ultrapower embedding. Note that κ is not measurable in M, since, else, κ would have been singular in $M[H]$, and therefore also in $V[G]$. Thus,

$$\kappa \in j_{U^*}\left(\{\xi < \kappa \colon \xi \text{ is not measurable in } V\}\right)$$

and thus $U = U^* \cap V$ concentrates on non-measurables. $\square$

Let us define the measure $U^\times \in V[G]$.

Definition 2.3. *Assume that $U \in V$ is a normal measure on κ. If U has Mitchell order 0, define $U^\times = U^*$. Assume otherwise. Let $d \colon \Delta \to \kappa$ be the function which maps every measurable cardinal of V to the first element in its Prikry sequence in $V[G]$. Set–*

$$U^\times = d_*\left(U^*\right) = \{A \subseteq \kappa \colon d^{-1}[A] \in U^*\}$$

We will prove that whenever U^* is non-normal, namely, $\Delta \in U^*$, d projects U^* to the normal measure below it in the Rudin-Keisler order; this projected measure is $U^\times$ defined above.

Lemma 2.4. *Let U be a normal measure on κ in V. Then $U^\times$ is a normal measure on κ in $V[G]$.*

Proof. We can assume that U has Mitchell order > 0. It suffices to prove that $[d]_{U^*} = \kappa$.

First, note that for every $x < \kappa$, $d^{-1}\{x\}$ is finite. Indeed, given an arbitrary condition $p \in P_\kappa$, let $b \subseteq \kappa$ be the finite set such for every $\xi \in \kappa \setminus b$, $p \restriction_\xi \geq^* 0_{\underset{\sim}{Q}_\xi}$. For every such ξ, let $p^* \geq^* p$ be such that x is removed from every measure one set. Then p^* forces that $d^{-1}\{x\}$ is finite, and since p was arbitrary, this indeed holds in $V[G]$.

This shows that $[d]_{W^*} \geq \kappa$. Assume that $f \in V[G]$ is a function in $V[G]$ such that, for every $\xi \in \Delta$, $f(\xi) < d(\xi)$. Let p be a condition which forces this. Assume that $q \geq p$ is arbitrary, and let ξ_0 be an ordinal which such that for every $\xi > \xi_0$, $q \restriction_\xi \Vdash q(\xi) \geq^* 0_{\underset{\sim}{Q}_\xi}$. For every $\xi \in \Delta$ above ξ_0, we describe a name for a subset of $P \setminus \xi$ which is forced by $q \restriction_\xi$ to be $\leq^*$ dense open subset of $P \setminus \xi$ above $q \setminus \xi$,

$$e(\xi) = \{r \geq q \setminus \xi \colon \text{there exists } \gamma < \xi \text{ such that } r \setminus \xi \Vdash \underset{\sim}{f}(\xi) = \gamma\}$$

The density follows since every name for an ordinal below the first element for a Prikry sequence can be decided by a direct extension.

By fusion, there exists $p^* \in G$ above p such that–

$$p^* \Vdash \exists \gamma < \kappa,\ \left(j_U(p) \setminus \kappa\right)^{-\kappa} \Vdash j_U\left(\underset{\sim}{f}\right)(\kappa) = \check{\gamma}$$

and by extending p^* to a condition in G, we can assume that it decides the value of $\gamma < \kappa$. So $\left(j_U(p^*)\right)^{-\kappa} \Vdash j_U\left(\underset{\sim}{f}\right)(\kappa) = \check{\gamma}$, and thus, in $V[G]$, $[f]_{U^*} = \gamma < \kappa$, as desired. $\square$

Claim 2.5. *Let $U \in V$ be a normal measure on κ. The following are equivalent:*

1. *U has Mitchell order 0 in V.*
2. *$U^\times = U^*$.*
3. *$d''\Delta \notin U^\times$.*

Proof. Clearly 1 implies 2 by the definition of $U^{\times}$.

Assume 2. If $d''\Delta \in U^{\times}$ then $d''\Delta \in U^{*}$, and thus, there exists $p \in G$ such that–

$$(j_U(p))^{-\kappa} \Vdash \check{\kappa} \in j_U(\underset{\sim}{d}''\Delta)$$

but this cannot happen, since $(j_U(p))^{-\kappa}$ forces that κ does not appear as an element in any of the Prikry sequences.

Finally, if U has Mitchell order higher than 0 in V, then $\Delta \in U$ and thus $\Delta \in U^{*}$. Therefore, $d''\Delta \in U^{\times}$. $\square$

Lemma 2.6. *Let U be a normal measure on κ in V with $o(U) > 0$. Let $j_{U^*}\colon V[G] \to M[H]$ be the ultrapower embedding of U^*. Then $[Id]_{U^*}$ is measurable in M, and κ appears as a first element in its Prikry sequence in $M[H]$. $[Id]_{U_*}$ is maximal with this property, namely, for every measurable above $[Id]_{U^*}$, κ does not appear in its Prikry sequence. Furthermore, for every $\mu > [Id]_{U^*}$ measurable in M, $d(\mu) > [Id]_{U^*}$.*

Proof. Since $\Delta \in U \subseteq U^{*}$, $[Id]_{U^*}$ is measurable in M. But–

$$\kappa = [d]_{U^*} = j_{U^*}(d)([Id]_{U^*})$$

so κ appears first in the Prikry sequence of $[Id]_{U^*}$ in $M[H]$.

Finally, fix any condition $p \in G$. Then–

$$(j_U(p))^{-\kappa} \Vdash \text{for every } \mu \in j_U(\Delta) \setminus (\kappa+1)\,,\ j_U(\underset{\sim}{d})(\mu) > \kappa$$

In particular, $\{\xi < \kappa\colon \text{for every } \mu \in \Delta \setminus (\xi+1)\,,\ d(\mu) > \xi\} \in U^{*}$. Thus, for every measurable $\mu > [Id]_{U^*}$, $d(\mu) > [Id]_{U^*}$. $\square$

Let us assume now that W is an arbitrary normal measure on κ in $V[G]$. Our goal will be to prove that $W = U^{\times}$ for some normal measure $U \in V$. Denote by $j_W\colon V[G] \to M[H]$ the ultrapower embedding of W over $V[G]$. We start with the following observation:

Claim 2.7. *Let W be a normal measure on κ in $V[G]$. Then–*

$$\kappa \setminus \bigcup_{\alpha \in \Delta} (d(\alpha), \alpha] \in W$$

Proof. [2] Assume otherwise. Then $X = \bigcup_{\alpha\in\Delta}(d(\alpha),\alpha] \in W$. We argue that there exists a regressive function $f\colon X \to \kappa$ which is not constant modulo W (this is a contradiction, since $W \in V[G]$ is normal, and hence all the sets in W are stationary in κ). Indeed, for every $\eta \in X$, let $\alpha_\eta \in \Delta$ be the first α such that $\eta \in (d(\alpha), \alpha]$. Then define $f(\eta) = d(\alpha_\eta)$. f is not constant modulo W since otherwise there exists $\xi < \kappa$ with $d^{-1}\{\xi\}$ infinite. $\square$

Remark 2.8. *$W \cap V$ is a normal measure in V of Mitchell order 0. Indeed, by corollay 1.7, $W \cap V \in V$. Clearly $W \cap V$ is normal in V. Finally, note that $\Delta \notin W \cap V$, namely $\kappa \notin j_W(\Delta)$. Otherwise, κ was measurable in M, and thus singular in $M[H] \subseteq V[G]$. But κ is regular in $V[G]$, a contradiction.*

Let us assume, by induction, that for every measurable $\mu < \kappa$, the normal measures on μ in V^{P_μ} have the form $U^{\times}$ for some normal measure U on μ in V. From the previous remark, we can assume also that every such $U^{\times}$ concentrates on non-measurables of V below μ.

[2]The proof presented here was offered by Omer Ben-Neria, and is a major simplification of the original argument.

Definition 2.9. *Let $W \in V[G]$ be a normal measure on κ. We now define a normal measure $W^* \in V[G]$ on κ. If $d''\Delta \notin W$, take $W^* = W$. Assume otherwise. For every $\delta < \kappa$, the set $d^{-1}\{\delta\}$ is finite (see the proof of lemma 2.4). Define a set $\Delta^* \subseteq \Delta$,*

$$\Delta^* = \{\xi \in \Delta \colon \xi = \max d^{-1}\{d(\xi)\}\}$$

Δ^* is an unbounded subset of Δ, on which d is injective. Let–

$$W^* = \{X \subseteq \kappa \colon d''(X \cap \Delta^*) \in W\}$$

W^* is a non-trivial, κ-complete ultrafilter on κ.

Let us review some of the properties of W^* in the case where $d''\Delta \in W$. Clearly $\Delta, \Delta^* \in W^*$. d is a Rudin-Keisler projection of W^* onto W, and is injective on $\Delta^* \in W^*$. Therefore $W \equiv_{RK} W^*$, and in particular $j_W = j_{W^*}$, namely W, W^* have the same ultrapower embedding from $V[G]$ to $M[H]$. In $M[H]$, $\kappa = [d]_{W^*} = j_{W^*}(d)([Id]_{W^*})$, namely κ is the first element in the Prikry sequence of $[Id]_{W^*}$. Finally, $\Delta^* \in W^*$, and thus–

$$[Id]_{W^*} = \max j_{W^*}(d)^{-1}\{\kappa\}$$

so κ does not appear as first element in the Prikry sequence of any measurable above $[Id]_{W^*}$.

Lemma 2.10. *$W^* \cap V \in V$ is a normal measure on κ in V.*

Proof. By corollary 1.7, $W^* \cap V \in V$. If $d''\Delta \notin W$, then $W^* = W$ is normal, and so is $W^* \cap V$. Let us assume that $d''\Delta \in W$. Assume that $f \in V$ and $\{\xi < \kappa \colon f(\xi) < \xi\} \in W^* \cap V$. Denote this set by A and assume that $A \subseteq \Delta$ (else, intersect).

For every $p \in P_\kappa$, there exists a direct extension $p^* \geq^* p$ and a finite subset $b \subseteq \kappa$ such that, for every $\xi \in A \setminus b$,

$$p^* \restriction_\xi \Vdash \underset{\sim}{A}_\xi^{p^*} \subseteq \xi \setminus (f(\xi)+1) \text{ and } t_\xi^{p^*} = \langle\rangle$$

thus, there exists such $b \subseteq \kappa$ and $p^* \in G$. Then p^* forces that for every $\xi \in A \setminus b$, $f(\xi) < d(\xi)$. But $A \in W^*$, and thus $A \setminus b \in W^*$, so, in $M[H]$, $[f]_{W^*} < [d]_{W^*} = d([Id]_{W^*}) = \kappa$. Therefore, there exists $\beta < \kappa$ such that–

$$\{\xi < \kappa \colon f(\xi) = \beta\} \in W^*$$

but this set belongs to V (since $f \in V$), and thus–

$$\{\xi < \kappa \colon f(\xi) = \beta\} \in W^* \cap V$$

as desired. □

Remark 2.11. *Given a normal measure on κ, $W \in V[G]$, we abuse the notation and denote by d the function $j_W(d) \colon j_W(\Delta) \to \kappa$. Similarly, given a normal measure $U \in V$ on κ, we use $\underset{\sim}{d}$ to denote the $j_U(P)$-name $j_U(\underset{\sim}{d})$.*

Lemma 2.12. *Let $p \in G$ be a condition. Then $(j_W(p))^{-[Id]_{W^*}} \in H$. In particular, if $d''\Delta \notin W$, Then $j_W(p)^{-\kappa} \in H$.*

Proof. $j_W(p) \in H$ since $p \in G$. In order to prove that $(j_W(p))^{-[Id]_{W^*}} \in H$, it suffices to prove that ordinals $\leq [Id]_{W^*}$ do not appear in Prikry sequences of measurables above $[Id]_{W^*}$ in $M[H]$.

Clearly, for every $\mu > [Id]_{W^*}$, $d(\mu) \geq \kappa$. Otherwise, there exist $\alpha < \kappa$ and $A \in W^*$ such that for every $\xi \in A$, there is some $\mu(\xi) > \xi$ with $d(\mu(\xi)) = \alpha$. In particular, $d^{-1}\{\alpha\}$ is infinite, a contradiction.

Let us argue now that for every $\mu > [Id]_{W^*}$, $d(\mu) > \kappa$. If $d''\Delta \notin W$ this is clear, since κ does not belong to the image of d in $M[H]$. Thus, let us take care of the case where $d''\Delta \in W$. In this case, recall that in $M[H]$, $[Id]_{W^*} = \max d^{-1}\{\kappa\}$. Thus, for every $\mu > [Id]_{W^*}$, $d(\mu) \neq \kappa$.

Finally, let us argue that for every $\mu > [Id]_{W^*}$, $d(\mu) > [Id]_{W^*}$. It suffices to prove that for every such μ, $d(\mu) \notin (\kappa, [Id]_{W^*}]$. If $d''\Delta \notin W$ this is clear, since in this case $W^* = W$ and $[Id]_{W^*} = \kappa$. Let us assume that $d''\Delta \in W$. We claim that in $V[G]$, there exists a finite set $b \subseteq \kappa$ such that for every measurable $\mu > \sup(b)$,

$$d(\mu) \notin \bigcup_{\xi \in \Delta \cap \mu} (d(\xi), \xi]$$

We prove this by a density argument. Fix a condition $p \in P_\kappa$. Let $b \subseteq \kappa$ be the set of coordinates such that for every $\mu > \sup(b)$, $p \restriction_\mu \Vdash p(\mu) \geq^* 0$. We extend p to $p^* \geq^* p$ such that, for every measurable $\mu > \sup(b)$,

$$p^* \restriction_\mu \Vdash \underset{\sim}{d}(\mu) \notin \bigcup_{\xi \in \Delta \cap \mu} (d(\xi), \xi]$$

this is possible since, by the induction hypothesis, the weakest condition in P_μ forces that–

$$\bigcup_{\xi \in \Delta \cap \mu} (d(\xi), \xi]$$

does not belong to any normal measure in V^{P_μ}. Pick such $p^* \in G$. Then in $V[G]$, for every $\mu > \sup(b)$,

$$d(\mu) \notin \bigcup_{\xi \in \Delta \cap \mu} (d(\xi), \xi]$$

This is true for every $\mu \in \Delta \setminus \sup(b) \in W^*$. Thus, in $M[H]$, for every $\mu > [Id]_{W^*}$, $d(\mu) \notin (\kappa, [Id]_{W^*}]$. $\square$

Proof of Theorem 0.1. Let $W \in V[G]$ be a normal measure on κ. Let $U = W^* \cap V$. Let $k \colon M_U \to M$ be the embedding which satisfies, for every $f \in V$,

$$k([f]_U) = [f]_{W^*}$$

It's not hard to verify that k is elementary and $j_W \restriction_V = j_{W^*} \restriction_V = k \circ j_U$. Moreover, $\operatorname{crit}(k) > \kappa$ if and only if $d''\Delta \notin W$: Indeed, if $d''\Delta \notin W$ then $W^* = W$ is normal and thus $k(\kappa) = \kappa$, and if $d''\Delta \in W$ then $\kappa = [d]_{W^*} < [Id]_{W^*} = k([Id]_U) = k(\kappa)$.
Let us argue now that $W = U^\times$.

Assume first that $d''\Delta \notin W$. Then Definition 2.9, $W^* = W$, and by Remark 2.8, U has Mitchell order 0. We argue that $W = U^\times = U^*$. It suffices to prove that $U^* \subseteq W$. Let $X \in U^*$, and assume that $\underset{\sim}{X} \in V$ is a P_κ-name such that $(\underset{\sim}{X})_G = X$. Then for some $p \in G$,

$$(j_U(p))^{-\kappa} \Vdash \check{\kappa} \in j_U(\underset{\sim}{X})$$

By applying $k \colon M_U \to M$,

$$(j_W(p))^{-\kappa} \Vdash \check{\kappa} \in j_W(\underset{\sim}{X})$$

where we used that fact that $k(\kappa) = \kappa$. Since $p \in G$ and $d''\Delta \notin W$, $(j_W(p))^{-\kappa} \in H$, and thus, in $M[H]$, $\kappa \in j_W(X)$, as desired.

Assume now that $d''\Delta \in W$. Then $\Delta \in W^*$ and thus $o(U) > 0$. In this case, $k(\kappa) = [Id]_{W^*} > \kappa$. Let us prove that $W = U^\times$. Since both are ultrafilters in $V[G]$, it suffices to prove that $U^\times \subseteq W$. Assume that $X \in U^\times$, and let $\underset{\sim}{X} \in V$ be such that $(\underset{\sim}{X})_G = X$. Let $p \in G$ be a condition such that–

$$(j_U(p))^{-\kappa} \Vdash \check{\kappa} \in j_U\left(\underset{\sim}{d}^{-1}\underset{\sim}{X}\right)$$

By applying $k\colon M_U \to M$,

$$(j_W(p))^{-[Id]_{W^*}} \Vdash \underset{\sim}{d}\left([Id]_{W^*}\right) \in j_W\left(\underset{\sim}{X}\right)$$

but $(j_W(p))^{-[Id]_{W^*}} \in H$ by lemma 2.12, and thus, in $M[H]$,

$$\kappa = d\left([Id]_{W^*}\right) \in \left(j_W\left(\underset{\sim}{X}\right)\right)_H = j_W(X)$$

so $X \in W$, as desired.

Finally, assume that $U \neq U'$ are normal measures in V. If both have Mitchell order 0, then $U^* \neq U'^*$ and thus $U^\times \neq U'^\times$. If exactly one of them, say U, has Mitchell order 0, then $d''\Delta \in U'^\times \setminus U^\times$. Thus, let us consider the case where both have Mitchell order higher than 0. Let $A \in U$, $B \in U'$ be disjoint sets. In $V[G]$, let $A^* = A \cap \Delta^* \in U^*$, $B^* = B \cap \Delta^* \in U'^*$. Then $d''A^* \in U^\times$, $d''B^* \in U'^\times$, and $d''A^* \cap d''B^* = \emptyset$ since d is injective on Δ^*. Thus $U^\times \neq U'^\times$. $\square$

The embedding $k\colon M_U \to M$ from the above proof will be used in the next sections to analyze the structure of $j_W \restriction_V$. For now, let us note that $\mathrm{crit}(k) = \kappa$ if and only if $d''\Delta \notin W$.

3 The Structure of $j_W \restriction_V$

Given a normal measure $W \in V[G]$ on κ, let $j_W\colon V[G] \to M[H]$ be the ultrapower embedding, and let $U \in V$ be a normal measure on κ such that $W = U^\times$. Our main goal in this section will be to factor $j_W \restriction_V$ to an iterated ultrapower of V.

We divide this section to several subsections. In the first subsection, we isolate a natural number $m < \omega$ and a sequence $U^0 \lhd U^1 \lhd \ldots \lhd U^m = U$ of measures on κ in V. In the second subsection, we describe in detail the structure of $j_W \restriction_V$ and sketch the main steps in the proof. We will also demonstrate the structure of $j_W \restriction_V$ in several simple cases. In the third subsection, we develop a generalization of the Fusion lemma. This generalization will be applied in the fourth subsection, where we complete the proof of theorem 0.2, provide a sufficient condition for the definability of $j_W \restriction_V$ in V, and describe the Prikry sequences added by H for measurables of M above κ. For instance, we will prove that each measure U^j, for $0 \leq j < m$, is iterated in $j_W \restriction_V$ ω-many times, producing Prikry sequences for one of the measurables in the finite set $d^{-1}\{\kappa\}$

The value of $m < \omega$ and the exact measures participating in the sequence $U^0 \lhd U^1 \lhd \ldots \lhd U^m = U$ depend on W and on the measures in the sequence $\langle W_\xi\colon \xi \in \Delta\rangle \in V[G]$, namely the measures used in G to singularize the measurables of Δ. For every $\xi \in \Delta$, denote by $U_\xi \in V$ the measure on ξ such that $W_\xi = U_\xi^\times$. By induction, for every $\xi \in \Delta$ there exists a natural number m_ξ and a sequence $U_\xi^0 \lhd \ldots \lhd U_\xi^{m_\xi - 1} \lhd U_\xi^{m_\xi} = U_\xi$ of normal measures on ξ in V. The identity of the measures $\langle U_\xi^i\colon \xi \in \Delta,\ j \leq m_\xi\rangle$ determines the measures participating in the iteration of $j_W \restriction_V$, and whether or not this iteration is definable in V.

3.1 The system $U^0 \lhd U^1 \lhd \ldots \lhd U^m$ associated with W

Denote $m = m(W) = \left|d^{-1}\{\kappa\}\right|$ as computed in $M[H]$. Namely, $m(W) < \omega$ is the number of occurrences of κ as a first element in Prikry sequences added to measurables in M. Possibly

$m(W) = 0$, in the case where $d''\Delta \notin W$. Define, for every $i \geq 1$, the set $\Delta_i \subseteq \Delta$:

$$\begin{aligned}\Delta_i =& \{\xi \in \Delta \colon \left|\xi \cap d^{-1}\{d(\xi)\}\right| = i - 1\} = \\ & \{\xi \in \Delta \colon \xi \text{ is the } i\text{-th element in } d^{-1}\{d(\xi)\}\}\end{aligned}$$

For $i = 0$, let $\Delta_0 = \kappa \setminus \Delta$, the set of non-measurables below κ. We state some straightforward properties:

Claim 3.1.

1. *$\{\xi < \kappa \colon \xi$ appears as first element in m Prikry sequences below $\kappa\} \in W$.*
2. *For all but finitely many $\xi \in \Delta$, if $m(W_\xi) = i - 1$ for some $1 \leq i < \omega$, then ξ is the i-th element in $d^{-1}\{d(\xi)\}$.*
3. *$d''\Delta_1 \supseteq d''\Delta_2 \supseteq \ldots \supseteq d''\Delta_n \supseteq \ldots$ ($n < \omega$).*
4. *m is the maximal index such that $d''\Delta_m \in W$.*

Note that d is injective on each of the sets Δ_i. Let us define, for every $1 \leq i \leq m$, a measure W^i as follows:

$$W^i = \{X \subseteq \kappa \colon d''(X \cap \Delta_i) \in W\}$$

In particular, W^m is the measure W^* defined in the previous section. Since d is injective on each set Δ_i,

$$W \equiv_{RK} W^1 \equiv_{RK} W^2 \equiv_{RK} \ldots \equiv_{RK} W^m = W^*$$

For every $1 \leq i \neq j \leq m$, let $\pi_{i,j} \colon \Delta_i \to \Delta_j$ be the function which maps each $\xi \in \Delta_i$ to the j-th element in $d^{-1}(d(\xi))$ (which typically exists. if not, set $\pi_{i,j}(\xi) = 0$). Then $\pi_{i,j}$, which projects W^i onto W^j, is injective on the set–

$$\{\xi \in \Delta_i \colon \left|d^{-1}(d(\xi))\right| = m\} \in W^i$$

Finally, denote, for every $1 \leq i \leq m$, $U^i = W^i \cap V \in V$, and note that $U = U^m$.
For sake of completeness, let us denote $W^0 = W$ and $U^0 = W \cap V$. By remark 2.8, U^0 concentrates on $\Delta_0 = \kappa \setminus \Delta$. We begin by studying the properties of U^0.

Lemma 3.2. *$U^0 \trianglelefteq U$ is a normal measure of Mitchell order 0 in V. $U^0 = U$ if and only if U already has Mitchell order 0 in V. Finally, if U has Mitchell order above 0 in V, then $U^0 = \{A \subseteq \kappa \colon \kappa \in k(A)\} \cap M_U$.*

We will need the following claim:

Claim 3.3. *Let $U \in V$ be a measure on κ. Then $M_U[G]$ and $V[G]$ have the same subsets of κ.*

Proof. First let us assume that U concentrates on non-measurables. We will then adjust the proof to the other case. Assume that $\underset{\sim}{A} \in V$ is a P-name for a subset of κ. For every non-measurable $\alpha < \kappa$, let $e(\alpha)$ be the $\leq^*$-dense open subset of $P \setminus \alpha$ which decides the value of $\underset{\sim}{A} \cap \alpha$ over V^{P_α}. By lemma 1.3, there exists $p \in G$ such that for every non-measurable $\alpha < \kappa$,

$$p \restriction_\alpha \Vdash (p \setminus \alpha)^{-\alpha} \in e(\alpha)$$

For every such α, let $\underset{\sim}{A}_\alpha \in V_\kappa$ be a P_α-name such that–

$$p \restriction_\alpha \Vdash p \setminus \alpha \Vdash \underset{\sim}{A} \cap \alpha = \underset{\sim}{A}_\alpha$$

The sequence $\langle \underset{\sim}{A}_\alpha \colon \alpha < \kappa\rangle$ belongs to M_U. Thus, $A = (\underset{\sim}{A})_G \in M_U[G]$, since–

$$A = \bigcup_{\alpha<\kappa} (\underset{\sim}{A}_\alpha)_{G_\alpha}$$

We now adjust the proof for the case where $\Delta \in U$. We apply Fusion as before. For every $\alpha \in \Delta$, let–

$$e(\alpha) = \{r \in P \setminus \alpha \colon \exists B \subseteq \alpha, r \Vdash \underset{\sim}{A} \cap \underset{\sim}{d}(\alpha) = B \cap \underset{\sim}{d}(\alpha)\}$$

Before proving that $e(\alpha)$ is indeed $\leq^*$-dense open, let us argue that this suffices. By Fusion, there exists $p \in G$, and, for every $\alpha \in \Delta$, a P_α-name $\underset{\sim}{B}_\alpha$ for a subset of α, such that for each such α,

$$p\restriction_\alpha \Vdash (p \setminus \alpha)^{-\alpha} \Vdash \underset{\sim}{A} \cap \underset{\sim}{d}(\alpha) = \underset{\sim}{B}_\alpha \cap \underset{\sim}{d}(\alpha)$$

By closure under κ-sequences, the sequence $\langle \underset{\sim}{B}_\alpha \colon \alpha \in \Delta\rangle$ belongs to M_U. Therefore, in $M_U[G]$, A can be constructed as follows:

$$A = \bigcup_{\alpha\in\Delta} \left((\underset{\sim}{B}_\alpha)_{G_\alpha} \cap d(\alpha)\right)$$

Let us prove now that $e(\alpha)$ is $\leq^*$-dense open. Pick $r \in P \setminus \alpha$. For every $\nu \in \underset{\sim}{A}^r_\alpha$, let $X_\nu \in W_\alpha = U^\times_\alpha$, $B_\nu \subseteq \nu$ and $s_\nu \geq^* r \setminus (\nu+1)$ be such that–

$$\langle \underset{\sim}{t}^{\,r}_\alpha{}^\frown\langle\nu\rangle, X_\nu\rangle^\frown s_\nu \Vdash \underset{\sim}{A} \cap \nu = B_\nu$$

This can be done since the direct extension order of $P \setminus \alpha$ is more than ν-closed. Now let $B = [\nu \mapsto B_\nu]_{W_\alpha}$. Pick $X \in W_\alpha$ such that for every $\nu \in X$, $B \cap \nu = B_\nu$.

Now direct extend r as follows: shrink $\underset{\sim}{A}^r_\alpha$ such that it is contained in $X \cap (\triangle_{\nu<\alpha} X_\nu)$. Then, direct extend $r \setminus (\alpha+1)$ to be $s_{d(\alpha)}$. Let $r^* \geq^* r$ be the condition obtained this way. Then $r^* \in e(\alpha)$ and this is witnessed by the set $B \subseteq \alpha$. □

Proof of Lemma 3.2. If U has Mitchell order 0 in V, then $W = U^\times = U^*$ and thus $U^0 = W \cap V = U$. Let us assume that U has Mitchell order higher than 0, namely $\Delta \in U$.

We provide a definition of U^0 which is different from the definition $U^0 = W \cap V$ as in the statement of the lemma. From the definition we provide, it will be simple to see that $U^0 \in M_U$. After that, we will prove that indeed $U^0 = W \cap V$.

In $V[G]$, define for every $\alpha \in \Delta$, $U^0_\alpha = W_\alpha \cap V \in V$. In V, let $\underset{\sim}{U}^0 = j_U\left(\langle \underset{\sim}{U}^0_\alpha \colon \alpha \in \Delta\rangle\right)(\kappa)$. This is a $j_U(P)\restriction_\kappa = P$-name for a normal measure of Mitchell order 0 which belongs to M_U. Let $U^0 = \left(\underset{\sim}{U}^0\right)_G \in M_U$. Then $U^0 \in V$ is a normal measure on κ of Mitchell order 0. Since $U^0 \lhd U$, it suffices to prove that $U^0 = W \cap V$. Assume that $A \in U^0$ holds, and consider this as a statement in $M_U[G]$. For some $p \in G$,

$$p \Vdash \check{A} \in j_U\left(\langle \underset{\sim}{U}^0_\alpha \colon \alpha \in \Delta\rangle\right)(\kappa)$$

Let $\alpha \mapsto A(\alpha)$ be a function in V which represents A in M_U. Then we can assume that for every $\alpha \in \Delta$,

$$p\restriction_\alpha \Vdash \check{A}(\alpha) \in \underset{\sim}{U}^0_\alpha \subseteq U^\times_\alpha$$

By lemma 1.3, there exists $p^* \geq^* p$ such that, for all but finitely $\alpha \in \Delta$,

$$(p^*)^{-\alpha} \Vdash \underset{\sim}{d}(\alpha) \in \check{A}(\alpha)$$

where $d(\alpha)$ is the first element in the Prikry sequence of α. Thus,

$$(j_U(p^*))^{-\kappa} \Vdash \check{\kappa} \in j_U\left(\underset{\sim}{d}^{-1}\left(\check{A}\right)\right)$$

and thus $d^{-1}(A) \in U^*$ in $V[G]$. Therefore,

$$d''\left(d^{-1}A\right) \in U^{\times} = W$$

so $A \in W$, as desired.

Finally, let us assume that $\Delta \in U$ and argue that $U^0 = \{A \subseteq \kappa \colon \kappa \in k(A)\} \cap M_U$. Since both are ultrafilters in M_U, it suffices to prove that $U^0 \subseteq \{A \subseteq \kappa \colon \kappa \in k(A)\} \cap M_U$.

Let $A \in U^0$ be a set, and assume that $\xi \mapsto A(\xi)$ is a function in V such that $[\xi \mapsto A(\xi)]_U = A$. Assume that $p \in G$ forces that $A \in \underset{\sim}{U}^0$. We can assume that for every $\xi < \kappa$, $p \restriction_{\xi} \Vdash A(\xi) \in U^0_{\xi}$, and in particular, $p \restriction_{\xi} \Vdash A(\xi) \in U^{\times}_{\xi}$.

Given any extension $q \geq p$ in P_κ, there exists $p^* \geq^* p$ and a finite subset $b \subseteq \kappa$ such that, for every $\xi \in \Delta \setminus b$,

$$p^* \restriction_{\xi} \Vdash \underset{\sim}{A}^{p^*}_{\xi} \subseteq A(\xi) \text{ and } t^{p^*}_{\xi} = \langle\rangle$$

and thus, there exists such $p^* \in G$. Since $\Delta \setminus b \in W^*$ and $j_{W^*}(p) \in H$, it follows that, in $M[H]$,

$$\kappa = d([Id]_{W^*}) \in [\xi \mapsto A_{\xi}]_{W^*} = k(A)$$

as desired. $\square$

Lemma 3.4. *For every $1 \leq i \leq m$, U^i is normal and has Mitchell order higher than 0. Furthermore,*

$$U^0 = U^0 \lhd U^1 \lhd U^2 \lhd \ldots \lhd U^m = U$$

Proof. The proof that each U^i is normal is identical to 2.10, and essentially follows from the fact that d projects each W^i onto W.

For $i \geq 1$, each U^i has Mitchell order above 0: otherwise, $\kappa \setminus \Delta \in U^i \subseteq W^i$, and this contradicts the fact that $\Delta_i \in W^i$ is disjoint from $\kappa \setminus \Delta$.

Let us prove that for every $1 \leq i < m$, $U^i \lhd U^{i+1}$. Work in $V[G]$. For every $\xi < \kappa$, let $U^{\times}_{\xi}$ be the normal measure used at stage ξ in the iteration. We define an ultrafilter U^i_{ξ}: if $U^{\times}_{\xi}$ concentrates on $d''(\Delta_i \cap \xi)$, set $\underset{\sim}{U}^i_{\xi}$ to be the ultrafilter which concentrates on $\Delta_i \cap \xi$ and is projected via d onto $U^{\times}_{\xi}$. Else, set $U^i_{\xi} = U^*_{\xi}$.

Let $\underset{\sim}{\mathcal{U}}^i \in V$ be the sequence of names for the measures U^i_{ξ} defined above. Consider in $M_{U^{i+1}}$ the P_κ-name $j_{U^{i+1}}\left(\underset{\sim}{\mathcal{U}}^i\right)(\kappa)$, and let–

$$F = \left(j_{U^{i+1}}\left(\underset{\sim}{\mathcal{U}}^i\right)(\kappa)\right)_G \cap M_{U^{i+1}} \in M_{U^{i+1}}$$

F is a normal measure on κ which belongs to $M_{U^{i+1}}$. Thus, it suffices to prove that $F = U^i$.

Pick $X \in F$. Let $p \in G$ be a condition such that $p \Vdash X \in j_{U^{i+1}}\left(\underset{\sim}{\mathcal{U}}^i\right)(\kappa)$, namely–

$$\{\xi \in \Delta \colon p \restriction_{\xi} \Vdash X \cap \xi \in U^i_{\xi}\} \in U^{i+1}$$

we would like to argue that U^i_{ξ} in the above equation is the measure which concentrates on Δ_i and is projected via d onto $U^{\times}_{\xi}$. This requires to have–

$$\{\xi \in \Delta \colon p \restriction_{\xi} \Vdash d''(\Delta_i \cap \xi) \in U^{\times}_{\xi}\} \in U^{i+1}$$

Let us argue that p can be extended inside G such that this holds. Work over $M_{U^{i+1}}$, and extend p in G such that–

$$p \parallel \Delta_i \in j_{U^{i+1}}\left(\underset{\sim}{U}^{\times}\right)(\kappa)$$

It's enough to argue that p decides the above statement in a positive way. Assume otherwise. Then–

$$\{\xi < \kappa \colon p \restriction_{\xi} \Vdash \Delta_i \cap \xi \notin U^{\times}_{\xi}\} \in U^{i+1} \subseteq W^{i+1}$$

For every ξ in the above set (but finitely many), $d(\xi) \notin d''\Delta_i$. In particular, W^{i+1} concentrates on such ξ-s, and thus in $M[H]$, $\kappa \notin d''\Delta_i$, which is a contradiction.

Thus we can assume that $p \in G$ and–

$$\{\xi \in \Delta \colon p \restriction_{\xi} \Vdash d''(X \cap \Delta_i \cap \xi) \in U^{\times}_{\xi}\} \in U^{i+1}$$

Therefore,

$$\{\xi \in \Delta \colon p \Vdash d(\xi) \in d''(X \cap \Delta_i \cap \xi)\} \in U^{i+1}$$

and thus, in $V[G]$,

$$\{\xi \in \Delta \colon d(\xi) \in d''(X \cap \Delta_i)\} \in W^{i+1}$$

so–

$$d''\{\xi \in \Delta \colon d(\xi) \in d''(X \cap \Delta_i)\} \in W$$

so $d''(X \cap \Delta_i) \in W$, and in particular, $X \in W^i$. So $X \in U^i = W^i \cap V$.

Finally, let us argue that $U^0 \lhd U^1$. Consider in M_{U^1} the name $j_{U^1}(\underset{\sim}{\mathcal{U}}^{\times})(\kappa)$, and let $F \in M_{U^1}$ be its value with respect to the generic G. It suffices to prove that $F = U^0$. given $X \in F$, there exists $p \in G$ such that–

$$\{\xi \in \Delta \colon p \restriction_{\xi} \Vdash X \cap \xi \in U^{\times}_{\xi}\} \in U^1$$

In $V[G]$,

$$\{\xi \in \Delta \colon d(\xi) \in X\} \in W^1$$

Recall that $W^1 \equiv_{RK} W$, and thus in $M[H]$,

$$d([Id]_{W^1}) \in j_{W^1}(X) = j_W(X) = k(j_U(X))$$

where $k \colon M_U \to M$ is the embedding which satisfies $k([f]_U) = [f]_{W^*}$. Recall that $\mathrm{crit}(k) = \kappa$, and thus $\kappa \in k(\kappa \cap j_U(X)) = k(X)$. In particular, $X \in U^0$. □

Remark 3.5. *Denote $\langle \mu_0^{*1}, \ldots, \mu_0^{*m}\rangle = d^{-1}\{\kappa\} = \langle [Id]_{W^1}, \ldots, [Id]_{W^*}\rangle$. Then for every $1 \leq i \leq m-1$, $U^i = \{X \subseteq \kappa \colon \mu_0^{*i} \in k(X)\}$. Indeed, assume that $X \subseteq \kappa$ and $\mu_0^{*i} \in k(X) = k(j_U(X) \cap \kappa) = j_{W^*}(X) \cap [Id]_{W^*}$. Since W^i and W^* are Rudin-Keisler equivalent and $\mu_0^{*i} = [Id]_{W^i}$, it follows that $X \in W^i$. Therefore $X \in U^i$.*

In M_U, we can derive a measure on $[\kappa]^m$ using k as follows:

$$\mathcal{E}_0 = \{X \subseteq [\kappa]^m : \langle \kappa, \mu_0^{*1}, \ldots, \mu_0^{*m-1}\rangle \in k(X)\}$$

Corollary 3.6. *$\mathcal{E}_0 \in M_U$ is the product measure $U^0 \times \ldots \times U^{m-1}$ on $[\kappa]^m$, namely, for every $X \subseteq [\kappa]^m$, $X \in \mathcal{E}_0$ of and only if–*

$$\{\nu_0 < \kappa \colon \{\nu_1 < \kappa \colon \ldots \{\nu_{m-1} < \kappa \colon \langle \nu_0, \ldots, \nu_{m-1}\rangle \in X\} \in U^{m-1} \ldots\} \in U^1\} \in U^0$$

Proof. It suffices to prove that for each $X \subseteq [\kappa]^m$, $X \in U^0 \times \ldots \times U^{m-1}$ implies that $X \in \mathcal{E}_0$. Indeed, given X in the product measure, there are sets $X_0 \in U^0, \ldots, X_{m-1} \in U^{m-1}$ such that–

$$(X_0 \times \ldots \times X_{m-1}) \cap [\kappa]^m \subseteq X$$

By Remark 3.5, it follows that–

$$\langle \mu_0^{*1}, \ldots, \mu_0^{*m}\rangle \in k(X_0) \times \ldots \times k(X_{m-1}) \subseteq k(X)$$

as desired. □

$\mathrm{Ult}(M_U, \mathcal{E}_0)$ is isomorphic to the finite iterated ultrapower of V, with decreasing order, with $U^0 \lhd U^1 \lhd \ldots \lhd U^{m-1} \lhd U^m$.

3.2 Description of the Iteration

Assume that $W \in V[G]$ is a normal measure on κ. Let $U \in V$ be a normal measure such that $W = U^{\times}$. Denote $\kappa^* = j_W(\kappa)$ (we will later prove that $\kappa^* = j_U(\kappa)$). Let $j_W : V[G] \to M[H]$ be the ultrapower embedding. We work by induction on $\alpha \le \kappa^*$ and define an iterated ultrapower $\langle M_\alpha \colon \alpha \le \kappa^* \rangle$. We define as well, for every $\alpha < \kappa^*$,

1. Elementary embeddings $j_\alpha \colon V \to M_\alpha$ and $k_\alpha \colon M_\alpha \to M$, such that $j_W \restriction_V = k_\alpha \circ j_\alpha$.
2. The ordinal $\mu_\alpha = \operatorname{crit}(k_\alpha)$, which will turn out to be measurable in M_α.
3. A natural number $1 \le m_\alpha < \omega$, and a sequence of normal measures on μ_α,
$$U^0_{\mu_\alpha} \lhd \ldots \lhd U^{m_\alpha - 1}_{\mu_\alpha}$$
each of them belong to M_α. We also denote by $\mathcal{E}_\alpha$ be the measure on $[\mu_\alpha]^{m_\alpha}$ defined by taking product of the above measures, namely, a set $X \subseteq [\mu_\alpha]^{m_\alpha}$ belongs to $\mathcal{E}_\alpha$ if and only if–
$$\{\nu_{m_{\alpha-1}} < \mu_\alpha \colon \{\ldots\{\nu_0 < \mu_\alpha \colon \langle \nu_0, \ldots, \nu_{m_\alpha - 1} \in X\rangle\} \in U^{m_\alpha-1}_{\mu_\alpha} \ldots\}\} \in U^0_{\mu_\alpha}$$
Possibly $m_\alpha = 1$ and then $\mathcal{E}_\alpha = U^0_{\mu_\alpha}$.

Let us demonstrate the first two steps in $j_W \restriction_V$. Recall the system $W \cap V = U^0 \lhd U^1 \lhd \ldots \lhd U^m = U$. First, let $M_0 = \operatorname{Ult}(V, U) = \operatorname{Ult}(V, U^m)$. Let $k_0 \colon M_0 \to M$ be the embedding which satisfies, for every $f \in V$,
$$k_0\left([f]_U\right) = [f]_{W^*}$$
k_0 is elementary since $U \subseteq W^*$; furthermore, $\mu_0 = \operatorname{crit}(k_0) = \kappa$. Assuming that $m = \left| j_W(d)^{-1}\{\kappa\} \right| \ge 1$, it turns out that $m_0 = m$ and $U^j_{\mu_0} = U^j$ for every $j \le m-1$. Thus, $\mathcal{E}_0$ is the product measure $U^0 \times \ldots \times U^{m-1}$ (as defined in the previous subsection). We will then define $M_1 = \operatorname{Ult}(M_0, \mathcal{E}_0)$. If $m = 0$, $\mu_0 = \operatorname{crit}(k_0)$ is the first measurable above κ in M_0, $m_0 = 1$ and $\mathcal{E}_0 = U^0_{\mu_0}$.

The iteration $\langle M_\alpha \colon \alpha \le \kappa^* \rangle$ is continuous, namely, for every limit $\alpha \le \kappa^*$, M_α is the direct limit of $\langle M_\beta \colon \beta < \alpha \rangle$. At successor steps, $M_{\alpha+1} = \operatorname{Ult}(M_\alpha, \mathcal{E}_\alpha)$.

For simplicity, we denote the sequence $[Id]_{\mathcal{E}_\alpha}$ by $[Id]_\alpha$. Arguing by induction, every element in M_α has the form–
$$j_\alpha(f)\left(j_{1,\alpha}(\kappa), j_{\alpha_0+1,\alpha}\left([Id]_{\alpha_0}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right) \tag{1}$$
for some $f \in V$ and $\alpha_0 < \ldots < \alpha_k < \alpha$.

Remark 3.7. *$M_{\alpha+1} = Ult(M_\alpha, \mathcal{E}_\alpha)$ can be viewed as iteration of length m_α of M_α, in the following sense: denote–*
$$M^{m_\alpha - 1}_\alpha = Ult\left(M_\alpha, U^{m_\alpha-1}_{\mu_\alpha}\right)$$
$$M^{m_\alpha - 2}_\alpha = Ult\left(M^{m_\alpha}_{\mu_\alpha}, U^{m_\alpha-2}_{\mu_\alpha}\right)$$
etc., up to–
$$M^0_\alpha = Ult\left(M^1_\alpha, U^0_{\mu_\alpha}\right)$$
and take $M_{\alpha+1} = M^0_\alpha$. Denote–
$$\mu^1_\alpha = j_{U^0_{\mu_\alpha}}(\mu_\alpha),\ \mu^2_\alpha = j_{U^1_{\mu_\alpha}}(\mu_\alpha),\ \ldots, \mu^{m_\alpha-1}_\alpha = j_{U^{m_\alpha-2}_{\mu_\alpha}}(\mu_\alpha)$$

Then each element in $M_{\alpha+1}$ has the form–

$$j_{\alpha,\alpha+1}(f)\left(\mu_\alpha, \mu_\alpha^1, \ldots, \mu_\alpha^{m_\alpha - 1}\right)$$

for some $f \in V$, and can be identified with $[f]_{\mathcal{E}_\alpha}$. In particular, if $Id\colon [\kappa]^{m_\alpha} \to V$ is the identity function, then–

$$[Id]_{\mathcal{E}_\alpha} = \langle \mu_\alpha, \mu_\alpha^1, \ldots, \mu_\alpha^{m_\alpha - 1} \rangle$$

Before we proceed, we would like to present several examples in the case where the Mitchell order is linear in V.

Example 1: Assume that the Mitchell order on each measurable is linear in V. For every $\alpha \in \Delta$, let $U_{\alpha,0}$ be the unique measure on α of order 0. Let $P = P_\kappa$ be the Magidor iteration, where, for each $\alpha \in \Delta$, the measure $U^*_{\alpha,0} = U^\times_{\alpha,0}$ is taken to be W_α. In $V[G]$, consider $W = U^*_{\kappa,0} = U^\times_{\kappa,0}$. In this case, $d''\Delta \notin W$, $m(W) = 0$ and $j_W \restriction_V$ is an iterated ultrapower of V, starting with $U_{\kappa,0}$. After this step κ is no longer measurable. Let $\alpha < \kappa^* = j_{U_{\kappa,0}}(\kappa)$. In M_α, μ_α is the least measurable $\geq \sup\{\mu_\beta \colon \beta < \alpha\}$ with cofinality above κ in V, and $M_{\alpha+1} = \mathrm{Ult}\left(M_\alpha, U^{M_\alpha}_{\mu_\alpha,0}\right)$ is the ultrapower with the unique measure of order 0 on μ_α in M_α.

Example 2: Assume the same settings as in the previous example, but now $W = U^\times$ for arbitrary U of order higher than 0 (below κ we still assume that measures of order 0 are used). We argue that now, $m = m(W) = 1$. First, since $o(U) > 0$, $d''\Delta \in W$, and thus $d^{-1}\{\kappa\} \neq \emptyset$ in $M[H]$. So $m \geq 1$. In order to prove that $m = 1$, it suffices to prove that the following property holds in $V[G]$: There exists a finite subset $b \subseteq \kappa$ such that d is an injection on $\Delta \setminus b$. Furthermore, other then finitely many, all the Prikry sequences G adds to measurables in Δ are pairwise disjoint. Let us provide the proof. For every $\alpha \in \Delta$, let $C_\alpha \subseteq \alpha$ be the Prikry sequence added to α in $V[G]$. Then, for every $\alpha \in \Delta$,

$$\bigcup_{\beta<\alpha} C_\beta \notin U^*_{\alpha,0} = U^\times_{\alpha,0} \tag{2}$$

since otherwise there exists $p \in G_\alpha$ such that $\left(j_{U_{\alpha,0}}(p)\right)^{-\alpha} \Vdash \alpha \in \bigcup_{\beta < j_{U_{\alpha,0}}(\alpha)} \underset{\sim}{C}_\beta$; but $\left(j_{U_{\alpha,0}}(p)\right)^{-\alpha}$ forces that α cannot belong to Prikry sequences of measurables above α, a contradiction.

Now we can apply equation 2 in a density argument: Every condition p can be direct extended to $p^* \geq^* p$ by removing from each set $\underset{\sim}{A}^p_\alpha \in W_\alpha$ (where $\alpha \in \Delta$) the set $\bigcup_{\beta<\alpha} C_\beta$. Then p^* forces that the Prikry sequences added to measurables of Δ, aside from finitely many, are pairwise disjoint.

Thus $m = 1$, and the system $U^0 \lhd U^1$ consists of $U_{\kappa,0} = U^0 \lhd U^1 = U$. The first step in $j_W \restriction_V$ is $\mathrm{Ult}(V, U)$, and $U_{\kappa,0}$ is applied ω-many times to produce a Prikry sequence of critical points to $[Id]_{W^1}$, which is the only element in $d^{-1}\{\kappa\}$. For every $\alpha < \kappa^*$, $M_{\alpha+1} = \mathrm{Ult}\left(M_\alpha, U^{M_\alpha}_{\mu_\alpha,0}\right)$ as in the previous example. The main difference is that the length of the iteration, $\kappa^* = j_W(\kappa) = j_U(\kappa)$ is strictly higher than $j_{U_{\kappa,0}}(\kappa)$.

Example 3: Assume again linearity of the Mitchell order, but now fix in advance $m < \omega$, and assume as well that $o(\kappa) = m + 1$, namely, the normal measures on κ are $U_{\kappa,0} \lhd U_{\kappa,1} \lhd \ldots \lhd U_{\kappa,m}$. We define the iteration $P = P_\kappa$ such that for every $\alpha \in \Delta$, the measure W_α is chosen as follows: If $o(\alpha) = l + 1$ for some $l \leq m$, use the measure $W_\alpha = U^\times_{\alpha,l}$. In $V[G]$, let $W = U^\times_{\kappa,m}$. We argue that $m(W) = m$. We work by induction: If $m = 0$, then $d''\Delta \notin W$ and thus $m(W) = 0$. Assume that $m \geq 1$. $W^* = U^*_{\kappa,m}$ concentrates on measurables $\alpha \in \Delta$ such that $W_\alpha = U^\times_{\alpha,m-1}$, and for each such α, $m(W_\alpha) = m-1$. So $W^* = U^*_{\kappa,m}$ concentrates on measurables $\alpha \in \Delta$ such that α is the m-th element in $d^{-1}\{d(\alpha)\}$, and thus $m(W) = m$. By linearity of the Mitchell order, the system $U^0 \lhd \ldots \lhd U^m$ is exactly the sequence $U_{\kappa,0} \lhd U_{\kappa,1} \lhd \ldots \lhd U_{\kappa,m}$.

$d^{-1}\{\kappa\} = \{[Id]_{W^1}, \ldots, [Id]_{W^m}\}$ contains exactly m elements, and each $[Id]_{W^i}$ (where $1 \leq i \leq m$) has Prikry sequence in $M[H]$ which is generated by iterating the measure $U_{\kappa,i-1}$ ω-many times.

We would like to define the embedding $k_\alpha \colon M_\alpha \to M$. We do this assuming that embeddings $k_\beta \colon M_\beta \to M$ have been defined for every $\beta < \alpha$. We also assume by induction that for each such $\beta < \alpha$, a sequence $\vec{\mu}^*_\beta = \langle \mu^{*0}_\beta, \ldots, \mu_\beta^{*m_\beta - 1}\rangle$ has been defined. We then define $k_\alpha \colon M_\alpha \to M$ as follows:

$$k_\alpha\left(j_\alpha(f)\left(j_{0,\alpha}(\kappa), j_{\alpha_0+1,\alpha}\left([Id]_{\alpha_0}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right)\right) = j_W(f)\left([Id]_{W^*}, \vec{\mu}^*_{\alpha_0}, \ldots, \vec{\mu}^*_{\alpha_k}\right)$$

for every $f \in V$ and $1 \leq \alpha_0 < \ldots < \alpha_k$.

We will prove by induction on $\alpha \leq \kappa^*$ that the following properties hold:

(A) $k_\alpha \colon M_\alpha \to M$ is elementary.

(B) Denote $\mu_\alpha = \mathrm{crit}(k_\alpha)$. Then μ_α is measurable in M_α. Moreover, μ_α is the least measurable $\mu \in M_\alpha$ which is greater or equal to $\sup\{\mu_\beta \colon \beta < \alpha\}$ and satisfies $(\mathrm{cf}(\mu))^V > \kappa$.

(C) Let $\mu^*_\alpha = k_\alpha(\mu_\alpha)$. Then μ_α appears as an element in the Prikry sequence of $k_\alpha(\mu_\alpha)$ in H. We will denote by t_α the initial segment of the Prikry sequence of $k_\alpha(\mu_\alpha)$ below μ_α, and by n_α the length of t_α.

(D) Let $\{\mu^{*1}_\alpha, \ldots, \mu^{*m_\alpha - 1}_\alpha\}$ be the increasing enumeration of $d^{-1}(\mu_\alpha)$ below μ^*_α, and denote as well $\mu^{*0} = \mu_\alpha, \mu^*_\alpha = \mu^{m_\alpha}_\alpha$ (possibly $m_\alpha = 1$ and then μ_α does not appear as first element in Prikry sequences of measurables below μ^*_α). For every $0 \leq j \leq m_\alpha$, there exists a measure $U^j_{\mu_\alpha} \in M_\alpha$ on μ_α, which satisfies–

$$k_\alpha\left(U^j_{\mu_\alpha}\right) = j_W\left(\xi \mapsto U^j_\xi\right)\left(k_\alpha(\mu_\alpha)\right)$$

Moreover,

$$U^0_{\mu_\alpha} \lhd U^1_{\mu_\alpha} \lhd \ldots \lhd U^{m_\alpha - 1}_{\mu_\alpha}$$

(E) The measure $\mathcal{E}_\alpha$ which corresponds to $U^0_{\mu_\alpha} \lhd U^1_{\mu_\alpha} \lhd \ldots \lhd U^{m_\alpha-1}_{\mu_\alpha}$ is derived from $k_\alpha \colon M_\alpha \to M$ in the following sense:

$$\mathcal{E}_\alpha = \{X \subseteq [\mu_\alpha]^{m_\alpha} : \langle \mu^{*0}_\alpha, \mu^{*1}_\alpha, \ldots, \mu^{*m_\alpha - 1}_\alpha\rangle \in k_\alpha(X)\} \cap M_\alpha$$

The proof of the above properties goes by induction on α. For $\alpha = 0$, $k_0 \colon M_0 = M_U \to M$ is the embedding which maps each $[f]_U$ to $[f]_{W^*}$; it has critical point $\mu_0 = \kappa$. In $M[H]$, μ_0 appears as a first element in the Prikry sequence of $k_0(\mu_0) = [Id]_{W^*}$, and of $m-1$ measurables ${\mu_0}^{*1} = [Id]_{W^1}, \ldots, {\mu_0}^{*m-1} = [Id]_{W^{m-1}}$. The measure $\mathcal{E}_0 \in M_0$ derived from k_0 using $\langle \mu_0, {\mu_0}^{*1}, \ldots, {\mu_0}^{*m-1}\rangle$ is indeed the product of $U^0 \lhd \ldots \lhd U^{m-1}$ by remark 3.5.

We proceed and prove the properties for arbitrary $0 < \alpha < \kappa^*$.

Lemma 3.8. *$k_\alpha \colon M_\alpha \to M$ is elementary, and $j_W \restriction_V = k_\alpha \circ j_\alpha$.*

Proof. For $\alpha = 0$, we already argued that $k_0 \colon M_0 \to M$ in elementary.

For simplicity, we will prove that for every $x, y \in M_\alpha$, $M_\alpha \vDash x \in y$ if and only if $M \vDash k_\alpha(x) \in k_\alpha(y)$.

Let us focus on the case where $\alpha = \alpha' + 1$ is successor, as the limit case is simpler. There are functions $f, g \in V$ and $\alpha_0 < \ldots < \alpha_k < \alpha'$ such that–

$$x = j_\alpha(f)\left(j_{0,\alpha}(\kappa), j_{\alpha_0+1,\alpha}\left([Id]_{\alpha_0}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right), j_{\alpha'+1,\alpha}\left([Id]_{\alpha'}\right)\right)$$

$$y = j_\alpha(g)\left(j_{0,\alpha}(\kappa), j_{\alpha_0+1,\alpha}\left([Id]_{\alpha_0}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right), j_{\alpha'+1,\alpha}\left([Id]_{\alpha'}\right)\right)$$

We assumed that $M_\alpha = \mathrm{Ult}\left(M_{\alpha'}, \mathcal{E}_{\alpha'}\right) \vDash x \in y$, namely,

$$\mathrm{Ult}\left(M_{\alpha'}, \mathcal{E}_{\alpha'}\right) \vDash [Id]_{\alpha'} \in j_{\alpha',\alpha}\left(X\right)$$

where X is the set–

$$\Big\{\vec{\xi}\colon\ j_{\alpha'}(f)\left(j_{0,\alpha'}(\kappa), j_{\alpha_0+1,\alpha'}\left([Id]_{\alpha_0}\right), \ldots, j_{\alpha_k+1,\alpha'}\left([Id]_{\alpha_k}\right), \vec{\xi}\right) \in$$
$$j_{\alpha'}(g)\left(j_{0,\alpha'}(\kappa), j_{\alpha_0+1,\alpha'}\left([Id]_{\alpha_0}\right), \ldots, j_{\alpha_k+1,\alpha'}\left([Id]_{\alpha_k}\right), \vec{\xi}\right)\Big\}$$

In particular, $X \in \mathcal{E}_{\alpha'}$, and thus $\vec{\mu}^*_{\alpha'} \in k_{\alpha'}(X)$. Since $j_W \restriction_V = k_{\alpha'} \circ j_{\alpha'}$, it follows that–

$$j_W\left(f\right)\left([Id]_{W^*}, \vec{\mu}^*_{\alpha_0}, \ldots, \vec{\mu}^*_{\alpha_k}, \vec{\mu}^*_{\alpha'}\right) \in j_W\left(g\right)\left([Id]_{W^*}, \vec{\mu}^*_{\alpha_0}, \ldots, \vec{\mu}^*_{\alpha_k}, \vec{\mu}^*_{\alpha'}\right)$$

namely $k_\alpha(x) \in k_\alpha(y)$. □

We will present the proof of properties (B)-(E) is the next subsections.

3.3 Multivariable Fusion

Assume from now on that $\alpha > 0$ is fixed, and we are at stage α in the inductive proof of properties (A)-(E). In this subsection we develop a generalization of lemma 1.3 (the Fusion lemma).

Since $\alpha > 0$, we may assume in equation 1 that $\alpha_0 = 0$. This will simplify some of the arguments below. We can also denote $\mathcal{E}_0^{'} = U^0 \times \ldots \times U^{m-1} \times U^m$ (including U^m, unlike $\mathcal{E}_0$) and $[Id]_0^{'} = [Id]_0{}^\frown j_{0,1}\left(\kappa\right)$ so that every element in M_α has the form–

$$j_\alpha\left(f\right)\left(j_{1,\alpha}\left([Id]_0^{'}\right), j_{\alpha_1+1,\alpha}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right) \tag{3}$$

for some $f \in V$ and $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k < \alpha$.

Definition 3.9. *Let $p \in P_\kappa$ be a condition and $m \geq 1$. We define, by induction, when an increasing sequence $\langle \xi, \xi^1, \ldots, \xi^m \rangle$ below κ is admissible for p. In case it is, we also define an extension $p^\frown\langle \xi, \xi^1, \ldots, \xi^m \rangle \geq p$.*
Intuitively, $\langle \xi, \xi^1, \ldots, \xi^m \rangle$ is admissible for p if p can be extended (in a specific way, described below) to a condition $p^\frown\langle \xi, \xi^1, \ldots, \xi^m \rangle$ which forces that $d^{-1}\{\xi\} = \{\xi^1, \ldots, \xi^m\}$.
We provide the definition under the assumption $p \geq^* 0$. Else, consider only sequences $\langle \xi, \xi^1, \ldots, \xi^m \rangle$ such that ξ is an upper bound of the finite set of the coordinates $\beta < \kappa$ in which $p(\beta)$ non-directly extends 0_{Q_β}.

1. *$\langle \xi, \xi^1 \rangle$ is admissible for p if $\left(p \restriction_{\xi^1}\right)^{-\xi} \Vdash \xi \in \underset{\sim}{A}^p_{\xi^1}$. In this case, we define–*

$$p^\frown\langle \xi, \xi^1 \rangle = \left(\left(p \restriction_{\xi^1}\right)^{-\xi}\right)^\frown \langle\langle \check{\xi} \rangle, \underset{\sim}{A}^p_{\xi^1}\rangle^\frown p \setminus \left(\xi^1 + 1\right)$$

2. *Assume that $1 \leq i \leq m-1$ and $\langle \xi, \xi^1, \ldots, \xi^i \rangle$ is admissible for p. Assume also that $q = p^\frown\langle \xi, \xi^1, \ldots, \xi^i \rangle$ has been defined. Then $\langle \xi, \xi^1, \ldots, \xi^i, \xi^{i+1} \rangle$ is admissible for p if $q^{-\xi_i} \restriction_{\xi^{i+1}} \Vdash \xi \in \underset{\sim}{A}^p_{\xi^{i+1}}$. In this case, we define–*

$$p^\frown\langle \xi, \xi^1, \ldots, \xi^i, \xi^{i+1} \rangle = \left(q^{-\xi_i} \restriction_{\xi^{i+1}}\right)^\frown \langle\langle \check{\xi} \rangle, \underset{\sim}{A}^p_{\xi^{i+1}}\rangle^\frown p \setminus \left(\xi^{i+1} + 1\right)$$

In the case where $i = m-1$, we make a minor change in the above definition and set–

$$p^\frown\langle \xi, \xi^1, \ldots, \xi^m \rangle = \left(q^{-\xi_{m-1}} \restriction_{\xi^m}\right)^\frown \langle\langle \check{\xi} \rangle, \underset{\sim}{A}^p_{\xi^m}\rangle^\frown \left(p \setminus \left(\xi^m + 1\right)\right)^{-\xi^m}$$

(namely, remove $\xi^m + 1$ from large sets in places above ξ^m).

In other words, if $\vec{\xi} = \langle \xi, \xi^1, \ldots, \xi^m \rangle$ is admissible for p, then we set–

$$
\begin{aligned}
p^\frown\langle\vec{\xi}\rangle = & \left(p \restriction_{\xi^1}\right)^{-\xi^\frown} \langle\langle\check{\xi}\rangle, \underset{\sim}{A}^p_{\xi^1}\rangle^\frown \left(p \restriction_{(\xi^1,\xi^2)}\right)^{-\xi^{1\frown}} \langle\langle\check{\xi}\rangle, \underset{\sim}{A}^p_{\xi^2}\rangle^\frown \ldots^\frown \\
& \left(p \restriction_{(\xi^{m-1},\xi^m)}\right)^{-\xi^{m-1\frown}} \langle\langle\check{\xi}\rangle, \underset{\sim}{A}^p_{\xi^m}\rangle^\frown \left(p \setminus (\xi^m+1)\right)^{-\xi^m}
\end{aligned}
$$

In $V[G]$, denote, for every $\xi \in d''\Delta$ with $\left|d^{-1}\{\xi\}\right| = m$, $d^{-1}\{\xi\} = \langle \mu_0^{*1}(\xi), \ldots, \mu_0^{*m}(\xi)\rangle = \vec{\mu}_0^*(\xi)$. Then in $M[H]$, the sequence $[\xi \mapsto \vec{\mu}_0^*(\xi)]_W$ is–

$$
\langle \kappa, \mu_0^{*1}, \ldots, \mu_0^{*m-1}, \mu_0^{*m}\rangle = \langle [Id]_W, [Id]_{W^1}, \ldots, [Id]_{W^{m-1}}, [Id]_{W^m}\rangle
$$

Theorem 3.10. *Let $p \in P_\kappa$. For every increasing $\vec{\xi} = \langle \xi, \xi^1, \ldots, \xi^m\rangle$, let $e\left(\vec{\xi}\right)$ be a P_ξ-name for a subset of $P \setminus \xi$ which is $\leq^*$-dense open above conditions which force that $d^{-1}\{\xi\} = \langle \xi^1, \ldots, \xi^m\rangle$. Then there exists $p^* \geq^* p$ and a set $X \in U^0 \times U^1 \times \ldots \times U^m$ such that for every increasing $\langle \xi, \xi^1, \ldots, \xi^m\rangle \in X$ which is admissible for p^*,*

$$
p^* \restriction_\xi \Vdash p^{*\frown}\langle \xi, \xi^1, \ldots, \xi^m\rangle \setminus \xi \in e\left(\xi, \xi^1, \ldots, \xi^m\right)
$$

Furthermore, if p^* as above is chosen in G, then $U^0 \times U^1 \times \ldots \times U^m$ concentrates on the set of admissible sequences for p^*, and–

$$
\begin{aligned}
\{\xi < \kappa \colon & \langle \xi, \mu_0^{*1}(\xi), \ldots, \mu_0^{*m}(\xi)\rangle \text{ is admissible for } p^*,\ p^{*\frown}\langle \xi, \mu_0^{*1}(\xi), \ldots, \mu_0^{*m}(\xi)\rangle \in G \\
& \text{and } p^* \restriction_\xi \Vdash p^{*\frown}\langle \xi, \mu_0^{*1}(\xi), \ldots, \mu_0^{*m}(\xi)\rangle \setminus \xi \in e\left(\xi, \mu_0^{*1}(\xi), \ldots, \mu_0^{*m}(\xi)\right)\} \in W
\end{aligned}
$$

Proof. Assume for simplicity that $p \geq^* 0$. Else, just work with values of ξ above some ordinal μ for which $p \setminus \mu \geq^* 0$.

Let us first sketch the main steps of the proof. We will first define, for every sequence $\vec{\xi} = \langle \xi, \xi^1, \ldots, \xi^m\rangle$, a condition $p\left(\vec{\xi}\right) = p\left(\xi, \xi^1, \ldots, \xi^m\right) \geq^* p$. We define it such that for every $1 \leq i < m$, if $\langle \xi, \xi^1, \ldots, \xi^i\rangle$ is admissible for $p\left(\vec{\xi}\right)$,

$$
p\left(\vec{\xi}\right)^\frown \langle \xi, \xi^1, \ldots, \xi^i\rangle \restriction_{\xi^{i+1}} \Vert\ \xi \in \underset{\sim}{A}^{p(\vec{\xi})}_{\xi^{i+1}}
$$

This can be done in a trivial way, by taking a direct extension which removes ξ from the measure one sets at the relevant coordinate; we will avoid such trivialities by shrinking the measure one sets only above $\xi + 1$ (namely, instead of shrinking a large set A to a set B, shrink it to $(A \cap (\xi+1)) \cup (B \setminus (\xi+1))$).

Once $p\left(\vec{\xi}\right)$ is defined, we define a condition $r\left(\vec{\xi}\right) \in P \setminus \xi$: If the sequence $\vec{\xi}$ is admissible for $p\left(\vec{\xi}\right)$, we take $r\left(\vec{\xi}\right) \geq^* \left(p\left(\vec{\xi}\right)^\frown \langle\vec{\xi}\rangle\right) \setminus \xi$, with $r\left(\vec{\xi}\right) \in e\left(\vec{\xi}\right)$. Else, take $r\left(\vec{\xi}\right) = p\left(\vec{\xi}\right) \setminus \xi$.

The second step will be to define, for every initial segment $\langle \xi, \xi^1, \ldots, \xi^i\rangle$ of $\langle \xi, \xi^1, \ldots, \xi^m\rangle$, a condition $r\left(\xi, \xi^1, \ldots, \xi^i\right) \in P \setminus \xi$, such that the family $\langle r\left(\xi, \xi^1, \ldots, \xi^i\right) : i \leq m\rangle$ is coherent in the following sense: There exists a set $X \in U^0 \times U^1 \times \ldots \times U^m$ such that, for every $\vec{\xi} \in X$ and for every $1 \leq i < j \leq m$,

$$
r\left(\xi, \xi^1, \ldots, \xi^i\right) \restriction_{\xi^{i+1}} = r\left(\xi, \xi^1, \ldots, \xi^j\right) \restriction_{\xi^{i+1}}
$$

The set X obtained in this step will be the set X from the formulation of the lemma. Since X belongs to the product measure, we can fix sets $X_0 \in U^0, \ldots, X_m \in U^m$ such that–

$$
(X_0 \times X_1 \times \ldots \times X_m) \cap [\kappa]^m \subseteq X
$$

The third step will be to plug together all the conditions $r\left(\xi,\xi^1,\ldots,\xi^i\right)$, $i\le m$. We will do this step by step, by constructing a sequence of direct extensions of the original condition p,

$$p\le^* p^0\le^* p^1\le^*\ldots\le^* p^m$$

where each p^i has the following property: For every increasing sequence $\langle\xi,\xi^1,\ldots,\xi^i\rangle\in X_0\times\ldots\times X_i$ which is admissible for p^i,

$$\left(\left(p^i\right)^\frown\langle\xi,\xi^1,\ldots,\xi^i\rangle\right)^{-\xi^i}\ge r\left(\xi,\xi^1,\ldots,\xi^i\right)$$

Eventually, the condition $p^*=p^m$ will be as required in the formulation of the theorem.

The fourth and final step will be the proof of the "furthermore" part in the formulation of the theorem.

Step 1: Construction of $p\left(\vec{\xi}\right)\in P$ and $r\left(\vec{\xi}\right)\in P\setminus\xi$. Fix a sequence $\vec{\xi}=\langle\xi,\xi^1,\ldots,\xi^m\rangle$. We construct $p\left(\vec{\xi}\right)\ge^* p$. Work in the forcing $P\restriction_{[\xi^{m-1},\xi^m)}$, above a generic extension for $P\restriction_{\xi^{m-1}}$ which contains $p\restriction_{\xi^{m-1}}$. We choose $p\left(\vec{\xi}\right)\restriction_{[\xi^{m-1},\xi^m)}\ge^* p\restriction_{[\xi^{m-1},\xi^m)}$ such that:

$$A^{p(\vec{\xi})}_{\xi^{m-1}}=\left(A^p_{\xi^{m-1}}\cap(\xi+1)\right)\cup\left(B\setminus(\xi+1)\right)$$

and–

$$p\left(\vec{\xi}\right)\restriction_{(\xi^{m-1},\xi^m)}=r$$

where $B\in W_{\xi^{m-1}}$ and $r\in P\restriction_{(\xi^{m-1},\xi^m)}$ are chosen such that $\langle\langle\xi\rangle,B\rangle^\frown r\parallel\xi\in\underset{\sim}{A}^p_{\xi^m}$ (in the forcing $P\restriction_{[\xi^{m-1},\xi^m)}$).

Now work in $P\restriction_{[\xi^{m-2},\xi^{m-1})}$. We choose $p\left(\vec{\xi}\right)\restriction_{[\xi^{m-2},\xi^{m-1})}\ge^* p\restriction_{[\xi^{m-2},\xi^{m-1})}$ in a similar manner:

$$A^{p(\vec{\xi})}_{\xi^{m-2}}=\left(A^p_{\xi^{m-2}}\cap(\xi+1)\right)\cup\left(B\setminus(\xi+1)\right)$$

and–

$$p\left(\vec{\xi}\right)\restriction_{(\xi^{m-1},\xi^m)}=r$$

where $B\in W_{\xi^{m-2}}$ and $r\in P\restriction_{(\xi^{m-2},\xi^{m-1})}$ are chosen such that $\langle\langle\xi\rangle,B\rangle^\frown r\parallel\xi\in\underset{\sim}{A}^p_{\xi^{m-1}}$, and also decide in which way $p\left(\vec{\xi}\right)\restriction_{[\xi^{m-1},\xi^m)}$ decides the statement $\xi\in\underset{\sim}{A}^p_{\xi^m}$.

Continue in this fashion, direct extending p in the intervals $\left[\xi^i,\xi^{i+1}\right)$, shrinking $\underset{\sim}{A}^p_{\xi^i}$ only above $\xi+1$, and deciding how $p\restriction_{[\xi^{i+1},\xi^j)}{}^\frown\langle\xi,\xi^{i+1},\ldots,\xi^j\rangle$ decides the statement $\xi\in\underset{\sim}{A}^p_{\xi^j}$, for every $j\ge i+1$; By our construction, it actually decides the statement $\xi\in\underset{\sim}{A}^{p(\vec{\xi})}_{\xi^j}$, since the large sets were shrinked only above $\xi+1$.

This produces the desired condition $p\left(\vec{\xi}\right)\ge^* p$. Now we define the conditions $r\left(\vec{\xi}\right)\in P\setminus\xi$ as above.

Step 2: Construction of the conditions $r\left(\xi,\xi^1,\ldots,\xi^j\right)\in P\setminus\xi$ for every $j\le m$. Given $j<m$ and $\langle\xi,\xi^1,\ldots,\xi^j\rangle$, let–

$$r\left(\xi,\xi^1,\ldots,\xi^j\right)=\left[\langle\xi^{j+1},\ldots,\xi^m\rangle\mapsto r\left(\xi,\xi^1,\ldots,\xi^j,\xi^{j+1},\ldots,\xi^m\right)\restriction_{\xi^{j+1}}\right]_{U^{j+1}\times\ldots\times U^m}$$

Fix such j and $\langle\xi,\xi^1,\ldots,\xi^j\rangle$. Since all the measures $U^k,k\le j$, are normal measures on κ, there are sets $X^j_{j+1}\left(\xi,\xi^1,\ldots,\xi^j\right)\in U^{j+1}$, $X^j_m\left(\xi,\xi^1,\ldots,\xi^j\right)\in U^m$ such that for every increasing sequence $\langle\xi^{j+1},\ldots,\xi^m\rangle\in X^j_{j+1}\left(\xi,\xi^1,\ldots,\xi^j\right)\times\ldots\times X^j_m\left(\xi,\xi^1,\ldots,\xi^j\right)$,

$$r\left(\xi,\xi^1,\ldots,\xi^j\right)\restriction_{\xi^{j+1}}=r\left(\xi,\xi^1,\ldots,\xi^m\right)\restriction_{\xi^{j+1}}$$

Define, for every $k \leq m$,

$$X_k = \bigcap_{j<k} \left(\underset{\langle \xi, \xi^1, \ldots, \xi^j \rangle}{\triangle} X_k^j \left(\xi, \xi^1, \ldots, \xi^j \right) \right) \in U^k$$

Namely, $\xi^k \in X_k$ if and only if, for every $j < k$ and increasing sequence $\langle \xi, \xi^1, \ldots, \xi^j \rangle$ below ξ^k, $\xi^k \in X_k^j \left(\xi, \xi^1, \ldots, \xi^j \right)$.

Then $X_0 \in U^0, \ldots, X_m \in U^m$ satisfy that for every $j \leq m$, and for every increasing sequence $\langle \xi, \xi^1, \ldots, \xi^m \rangle \in (X_0 \times \ldots \times X_m) \cap [\kappa]^m$,

$$\langle \xi^{j+1}, \ldots, \xi^m \rangle \in X_{j+1}^j \left(\xi, \xi^1, \ldots, \xi^j \right) \times \ldots \times X_m^j \left(\xi, \xi^1, \ldots, \xi^j \right)$$

and thus–

$$r \left(\xi, \xi^1, \ldots, \xi^j \right) \restriction_{\xi^{j+1}} = r \left(\xi, \xi^1, \ldots, \xi^m \right) \restriction_{\xi^{j+1}}$$

as desired.

Step 3: Construction of the sequence $p \leq^* p^0 \leq^* \ldots \leq^* p^m$.

We first construct $p^0 \geq^* p$. Recall that for every $\xi \in X_0$, a condition $r(\xi) \in P \setminus \xi$ is defined such that $r(\xi) \geq^* p \setminus \xi$. By the Fusion lemma 1.3, we can choose $p^0 \geq^* p$ such that for every $\xi \in X_0$, $p^0 \restriction_\xi \Vdash p^0 \setminus \xi \geq^* q(\xi)$.

Assume that $i < m$ and p^i has been constructed such that, for every $\langle \xi, \xi^1, \ldots, \xi^i \rangle \in X_0 \times \ldots \times X_i$ which is admissible to it,

$$p^i \restriction_\xi \Vdash \left((p^i)^\frown \langle \xi, \xi^1, \ldots, \xi^i \rangle \setminus \xi \right)^{-\xi^i} \geq r \left(\xi, \xi^1, \ldots, \xi^i \right)$$

We now construct $p^{i+1} \geq^* p^i$. We will define, for every $\xi^{i+1} \in X_{i+1}$, a direct extension $q\left(\xi^{i+1}\right) \geq^* p^i \setminus \xi^{i+1}$. $p^{i+1} \geq^* p^i$ will be generated from the conditions $\langle q\left(\xi^{i+1}\right) : \xi^{i+1} \in X_{i+1} \rangle$ using the Fusion lemma 1.3. Fix $\xi^{i+1} \in X_{i+1}$ and work in the quotient forcing $P \setminus \xi^{i+1}$. For every $\xi \in A^{p^i}_{\xi^{i+1}} \cap X_0$, we define a direct extension–

$$\langle \langle \xi \rangle, B_\xi \rangle^\frown s_\xi \geq^* \langle \langle \xi \rangle, A^{p^i}_{\xi^{i+1}} \setminus (\xi + 1) \rangle^\frown p^i \setminus \left(\xi^{i+1} + 1 \right)$$

such that, if–

1. The increasing enumeration of $d^{-1}\{\xi\} \cap \xi^{i+1}$ is a sequence $\langle \xi^1, \ldots, \xi^i \rangle$ of length i;
2. $\langle \xi^1, \ldots, \xi^i \rangle \in X_1 \times \ldots \times X_i$;
3. $r\left(\xi, \xi^1, \ldots, \xi^i, \xi^{i+1}\right) \restriction_{\xi^{i+1}}$ belongs to the generic extension up to coordinate ξ^{i+1};

then $\langle \langle \xi \rangle, B_\xi \rangle^\frown s_\xi \geq r\left(\xi, \xi^1, \ldots, \xi^i, \xi^{i+1}\right) \setminus \xi^{i+1}$. Such B_ξ, r_ξ can be chosen since $r\left(\xi, \xi^1, \ldots, \xi^{i+1}\right) \setminus \xi^{i+1}$ is an extension of $p \setminus \xi^{i+1}$ which is obtained by direct extending after appending ξ to $t^p_{\xi^{i+1}}$ (if possible). Finally, take–

$$q(\xi^{i+1}) = \langle \langle \rangle, \underset{\sim}{A}^{p^i}_{\xi^{i+1}} \cap \left(\underset{\xi < \xi^{i+1}}{\triangle} B_\xi \right) \rangle^\frown s_{d(\xi^{i+1})}$$

This concludes the construction of p^{i+1}. Let us show that for every $\langle \xi, \xi^1, \ldots, \xi^{i+1} \rangle \in X_0 \times \ldots \times X_{i+1}$ which is admissible for p^{i+1},

$$p^{i+1} \restriction_\xi \Vdash \left((p^{i+1})^\frown \langle \xi, \xi^1, \ldots, \xi^{i+1} \rangle \setminus \xi \right)^{-\xi^{i+1}} \geq r \left(\xi, \xi^1, \ldots, \xi^{i+1} \right)$$

Pick such a sequence $\langle \xi, \xi^1, \ldots, \xi^i, \xi^{i+1} \rangle$. By induction, since $p^{i+1} \geq^* p^i$ and $\langle \xi, \xi^1, \ldots, \xi^i \rangle \in X_0 \times \ldots \times X_i$ is admissible for p^{i+1},

$$p^{i+1} \restriction_{\xi} \Vdash \left(\left(p^{i+1}\right)^\frown \langle \xi, \xi^1, \ldots, \xi^i \rangle \restriction_{[\xi, \xi^{i+1})} \right)^{-\xi^i} \geq r\left(\xi, \xi^1, \ldots, \xi^i\right) \restriction_{\xi^{i+1}} = r\left(\xi, \xi^1, \ldots, \xi^i, \xi^{i+1}\right) \restriction_{\xi^{i+1}}$$

Now, work in a generic extension $G_{\xi^{i+1}} \subseteq P_{\xi^{i+1}}$ which contains $\left(p^{i+1}\right)^\frown \langle \xi, \xi^1, \ldots, \xi^{i+1} \rangle \restriction_{\xi^{i+1}}$. By fusion, $p^{i+1} \restriction_{\xi^{i+1}}$ forces that $\left(p^{i+1} \setminus \xi^{i+1}\right)^{-\xi^{i+1}}$ extends $q\left(\xi^{i+1}\right)$. By the above formula, $r\left(\xi, \xi^1, \ldots, \xi^{i+1}\right) \restriction_{\xi^{i+1}} \in G_{\xi^{i+1}}$. Thus, by the choice of $q\left(\xi^{i+1}\right)$,

$$\left(\langle \langle \xi \rangle, A^{p^{i+1}} \setminus \xi + 1 \rangle^\frown p^{i+1} \setminus \left(\xi^{i+1} + 1\right) \right)^{-\xi^{i+1}} \geq r\left(\xi, \xi^1, \ldots, \xi^{i+1}\right) \setminus \xi^{i+1}$$

This is true for every generic $G_{\xi^{i+1}}$ which contains $\left(p^{i+1}\right)^\frown \langle \xi, \xi^1, \ldots, \xi^{i+1} \rangle \restriction_{\xi^{i+1}}$, so–

$$p^{i+1} \restriction_{\xi} \Vdash \left(\left(p^{i+1}\right)^\frown \langle \xi, \xi^1, \ldots, \xi^{i+1} \rangle \setminus \xi \right)^{-\xi^{i+1}} \geq r\left(\xi, \xi^1, \ldots, \xi^{i+1}\right)$$

as desired.

Step 4: The "furthermore" part in the formulation of the theorem. Denote $p^* = p^m$ and assume that $p^* \in G$. p^* satisfies that for every increasing sequence $\langle \xi, \xi^1, \ldots, \xi^m \rangle \in X_0 \times \ldots \times X_m$, and for every $1 \leq i \leq m$,

$$\left(p^{*\frown} \langle \vec{\xi} \rangle \right)^{-\xi^i} \restriction_{\xi^i} \| \; \xi \in \underset{\sim}{A}^{p^*}_{\xi^i} \tag{4}$$

Let us argue that–

$$\{ \xi < \kappa \colon \langle \vec{\mu}^*_0(\xi) \rangle \text{ is admissible for } p^* \} \in W$$

Take $X \in W$ such that $X \subseteq X_0 \cap d'' X_1 \cap \ldots \cap d'' X_m$. For every $\xi \in X$, $\vec{\mu}^*_0(\xi) \in X_0 \times X_1 \times \ldots \times X_m$. Then X can be shrinked to a set in W for which the decisions in equation 4 are positive when substituting $\vec{\xi} = \vec{\mu}^*_0(\xi)$: Indeed, otherwise, in $M[H]$, it would not hold that $d^{-1}\{\kappa\} = \langle \mu^{*1}_0, \ldots, \mu^{*m}_0 \rangle$.

Let us verify that $\{ \xi < \kappa \colon p^{*\frown} \langle \vec{\mu}^*_0(\xi) \rangle \in G \} \in W$. We need to verify that for a set of ξ-s in W the following holds: for every $1 \leq i \leq m$ and for every measurable $\mu \in \left(\mu^{*i-1}(\xi), \mu^{*i}(\xi) \right)$, $d(\mu) > \mu^{*i-1}(\xi)$.

Recall the following property from the proof of lemma 2.12: If $d''\Delta \in W$ (namely $m > 0$; if $m = 0$ there is nothing to prove), then there exists a finite subset $b \subseteq \kappa$ such that for every measurable $\mu > \sup(b)$,

$$d(\mu) \notin \bigcup_{\xi \in \Delta \cap \mu} (d(\xi), \xi]$$

from now on, we consider values of ξ above $\sup(b)$, such that $d^{-1}\{\xi\}$ contains only measurables μ for which the above holds (the set of such ξ-s is clearly in W). Let $\mu \in \left(\mu^{*i-1}(\xi), \mu^{*i}(\xi) \right)$. First, note that $\mu < \mu^{*i}_{\xi}$, and thus–

$$\xi = d\left(\mu^{*i}(\xi) \right) \notin (d(\mu), \mu]$$

so $d(\mu) \geq \xi$, and thus $d(\mu) > \xi$. This also proves the desired property for $i = 1$. Assume now that $i > 1$. Then $\mu > \mu^{*i-1}(\xi)$ and thus–

$$d(\mu) \notin \left(d\left(\mu^{*i-1}(\xi) \right), \mu^{*i-1}(\xi) \right] = \left(\xi, \mu^{*i-1}(\xi) \right]$$

since we already proved that $d(\mu) > \xi$, it follows that $d(\mu) > \mu^{*i-1}(\xi)$, as desired. $\square$

Recall that, given $\beta < \alpha$, μ_β appears as an element in the Prikry sequence of $k_\beta(\mu_\beta)$. Also, t_β is the initial segment of this sequence, consisting of all the ordinals below μ_β; we also denote $n_\beta = \mathrm{lh}(t_\beta)$. Finally, there exists a natural number $m_\beta < \omega$ and a corresponding sequence of measures,

$$U^0_{\mu_\beta} \lhd U^1_{\mu_\beta} \lhd \ldots \lhd U^{m_\beta - 1}_{\mu_\beta}$$

each of them belong to M_β.

We would like to construct, in V, functions which represent $\mu_\beta, t_\beta, U^j_{\mu_\beta}$ $(0 \leq j < m_\beta)$ in $\mathrm{Ult}(V[G], W)$. To do this, we first need to understand how the same objects are represented in the iterated ultrapower $j_\beta \colon V \to M_\beta$, in the sense of the following definition.

Definition 3.11. *Fix $0 < \alpha \leq \kappa^*$. An increasing sequence $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k$ below α is called nice, if the are functions g_i, f_i, F^j_i for every $1 \leq i \leq k$ and $0 \leq j \leq m_{\alpha_i}$ such that–*

$$\mu_{\alpha_1} = j_{\alpha_1}(g_1)\left(j_{1,\alpha_1}\left([Id]'_0\right)\right)$$

$$t_{\alpha_1} = j_{\alpha_1}(f_1)\left(j_{1,\alpha_1}\left([Id]'_0\right)\right)$$

$$U^j_{\mu_{\alpha_1}} = j_{\alpha_1}\left(F^j_1\right)\left(j_{1,\alpha_1}\left([Id]'_0\right)\right) \quad (0 \leq j < m_1)$$

and, for every $1 \leq i < k$,

$$\mu_{\alpha_{i+1}} = j_{\alpha_{i+1}}(g_{i+1})\left(j_{1,\alpha_1}\left([Id]'_0\right), j_{\alpha_1,\alpha_{i+1}}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_i,\alpha_{i+1}}\left([Id]_{\alpha_i}\right)\right)$$

$$t_{\alpha_{i+1}} = j_{\alpha_{i+1}}(f_{i+1})\left(j_{1,\alpha_1}\left([Id]'_0\right), j_{\alpha_1,\alpha_{i+1}}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_i,\alpha_{i+1}}\left([Id]_{\alpha_i}\right)\right)$$

$$U^j_{\mu_{\alpha_{i+1}}} = j_{\alpha_{i+1}}\left(F^j_{i+1}\right)\left(j_{1,\alpha_1}\left([Id]'_0\right), j_{\alpha_1,\alpha_{i+1}}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_i,\alpha_{i+1}}\left([Id]_{\alpha_i}\right)\right) \quad (0 \leq j < m_{\alpha_{i+1}})$$

Finally, denote by n_i the length of the sequence t_{α_i}.

The main application of nice sequences is to construct functions representing the cardinals in the sequences $\vec{\mu}^*_{\alpha_i}$ in $\mathrm{Ult}(V[G], W)$, using only functions in V and partial information about Prikry sequences added in $V[G]$. We demonstrate this, working by induction.

1. For every $\xi < \kappa$ with $\left|d^{-1}(\xi)\right| = m$, recall the sequence $\langle \mu^{*1}_0(\xi), \ldots, \mu^{*m}_0(\xi)\rangle$ which is the increasing enumeration of $d^{-1}(\xi)$. Denote–

$$\vec{\mu}^*_0(\xi) = \langle \xi, \mu^{*1}_0(\xi), \ldots, \mu^{*m}_0(\xi)\rangle$$

   Then in $M[H]$, the sequence $[\xi \mapsto \vec{\mu}^*_0(\xi)]_W$ is–

$$\langle \kappa = \mu_0, \mu^{*1}_0, \ldots, \mu^{*m-1}_0, \mu^{*m}_0\rangle = \langle [Id]_W, [Id]_{W^1}, \ldots, [Id]_{W^{m-1}}, [Id]_{W^m}\rangle$$

2. Given $\xi < \kappa$, let $\mu_{\alpha_1}(\xi)$ be the (n_1+1)-th element in the Prikry sequence of $g_1 = g_1(\vec{\mu}^*_0(\xi))$ (typically, this is the element which appears in this sequence after the initial segment $f_1(\vec{\mu}^*_0(\xi))$, which represents t_{α_1} in $\mathrm{Ult}(V[G], W)$). Let $\langle \mu^{*1}_{\alpha_1}(\xi), \ldots, \mu^{*m_{\alpha_1}-1}_{\alpha_1}(\xi)\rangle$ be the increasing enumeration of $d^{-1}(\mu_{\alpha_1}(\xi))$ below g_1. Denote $\mu^{*m_{\alpha_1}}_{\alpha_1}(\xi) = g_1$ and–

$$\vec{\mu}^*_{\alpha_1}(\xi) = \langle \mu_{\alpha_1}(\xi), \mu^{*1}_{\alpha_1}(\xi), \ldots, \mu^{*m_1-1}_{\alpha_1}(\xi)\rangle$$

   Then in $M[H] \simeq \mathrm{Ult}(V[G], W)$,

$$\left[\xi \mapsto \vec{\mu}^*_{\alpha_1}(\xi)\right]_W = \langle \mu_{\alpha_1}, \mu^{*1}_{\alpha_1}, \ldots, \mu^{*m_1-1}_{\alpha_1}\rangle$$

namely, this sequence starts with μ_{α_1}, concatenated with the increasing enumeration of $d^{-1}\{\mu_{\alpha_1}\}$ in $M[H]$. Let us verify this. Assume for simplicity that t_{α_1} is empty, namely $n_1 = 0$, or, in other words, μ_{α_1} is the first element in the Prikry sequence of $k_{\alpha_1}(\mu_{\alpha_1})$ (the fact that μ_{α_1} appears in this Prikry sequence, follows from property (C) of k_{α_1}). The first element in $\left[\xi \mapsto \vec{\mu}^*_{\alpha_1}(\xi)\right]_W$ is–

$$d\left([\xi \mapsto g_1(\vec{\mu}^*_0(\xi))]_W\right) = d\left(j_W(g_1)\left([Id]_W, [Id]_{W^1}, \ldots, [Id]_{W^m}\right)\right) = d\left(k_{\alpha_1}(\mu_{\alpha_1})\right) = \mu_{\alpha_1}$$

From this it is implied that the rest of the elements in $\left[\xi \mapsto \vec{\mu}^*_{\alpha_1}(\xi)\right]_W$ are the increasing enumeration of $d^{-1}\{\mu_{\alpha_1}\}$ below $k_{\alpha_1}(\mu_{\alpha_1})$, which is exactly $\langle \mu^{*1}_{\alpha_1}, \ldots, \mu^{*m_{\alpha_1}-1}_{\alpha_1}\rangle$.

3. Assuming that $0 \leq j < k$ and the functions $\vec{\mu}^*_{\alpha_0}(\xi), \ldots, \vec{\mu}^*_{\alpha_j}(\xi)$ have been defined, let $\mu_{\alpha_{j+1}}(\xi)$ be the n_{j+1}-th element in the Prikry sequence of $g_{j+1}\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_j}(\xi)\right)$ in $M[H]$. Let $\langle \mu^{*1}_{\alpha_{j+1}}(\xi), \ldots, \mu^{*m_{j+1}-1}_{\alpha_{j+1}}(\xi)\rangle$ be the increasing enumeration of $d^{-1}\left(\mu_{\alpha_{j+1}}(\xi)\right)$ below–
$$\mu^{*m_{j+1}}_{\alpha_{j+1}}(\xi) = g_{j+1}\left(\vec{\xi}, \vec{\mu}^*_{\alpha_0}(\xi), \ldots, \vec{\mu}^*_{\alpha_j}(\xi)\right)$$
Also, denote–
$$\vec{\mu}^*_{\alpha_{j+1}}(\xi) = \langle \mu^{*1}_{\alpha_{j+1}}(\xi), \ldots, \mu^{*m_{j+1}-1}_{\alpha_{j+1}}(\xi)\rangle$$
Then in $M[H] \simeq \mathrm{Ult}(V[G], W)$,
$$\left[\xi \mapsto \vec{\mu}^*_{\alpha_{j+1}}(\xi)\right]_W = \langle \mu_{\alpha_{j+1}}, \mu^{*1}_{\alpha_{j+1}}, \ldots, \mu^{*m_{j+1}-1}_{\alpha_{j+1}}\rangle$$
namely, this sequence starts with $\mu_{\alpha_{j+1}}$, concatenated with the increasing enumeration of $d^{-1}\{\mu_{\alpha_{j+1}}\}$ in $M[H]$. This is proved similarly to the previous point.

Denote $\mu_\alpha = \mathrm{crit}(k_\alpha)$. Write $\mu_\alpha = j_\alpha(h)(\kappa, \mu_{\alpha_0}, \ldots, \mu_{\alpha_k})$, where $\alpha_0 < \ldots < \alpha_k < \alpha$ is a nice sequence. Let $m_0, \ldots, m_k$ be such that $m_i = m_{\alpha_i}$. Denote $m = m_0$. Let g_i, f_i, F^j_i be functions as above.

Note that, by induction, $\left[\xi \mapsto F^j_{i+1}(\xi, \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_i}(\xi))\right]_W = \left[\xi \mapsto U^j_{\mu_{\alpha_{i+1}}(\xi)}\right]_W$ for every $0 \leq i \leq k$ and $0 \leq j \leq m_{\alpha_{i+1}}$ (Recall that, for a measurable $\eta \in \Delta$, U^j_η is the j-th measure in the system $U^0_\eta \lhd \ldots \lhd U^{m_\eta}_\eta$ associated with η). Thus, for a set of ξ-s in W,

$$F^j_{i+1}(\xi, \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_i}(\xi)) = U^j_{\mu_{\alpha_{i+1}}(\xi)}$$

Definition 3.12. *Fix a nice sequence $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_k$ below α. Given a condition $p \in P_\kappa$ and a sequence of increasing sequences–*

$$\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle = \langle\langle \xi, \xi^1, \ldots, \xi^m\rangle, \langle \nu_1, \nu^1_1, \ldots, \nu^{m_0-1}_1\rangle, \langle \nu_2, \nu^1_2, \ldots, \nu^{m_1-1}_2\rangle, \ldots\ldots, \langle \nu_k, \nu^1_k, \ldots, \nu^{m_k-1}_k\rangle\rangle$$

we define whenever $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle$ is admissible for p, and in this case, we define an extension $p^\frown\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle \geq p$.

1. *An increasing sequence $\vec{\xi} = \langle \xi, \xi^1, \ldots, \xi^m\rangle$ is admissible for p if it is admissible for p in the sense of definition 3.9. If it is, the extension $p^\frown\langle \xi, \xi^1, \ldots, \xi^m\rangle$ is defined the same as in 3.9.*

2. *Let $1 \leq i < k$. Assume that $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i \rangle$ is admissible for p and $q = p^\frown \langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i \rangle$ has been defined. Denote–*

$$g_{i+1} = g_{i+1}\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\right)$$

$$t_{i+1} = f_{i+1}\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\right)$$

$$F^j_{i+1} = F^j_{i+1}\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\right) \ \ (0 \leq j < m_{\alpha_{i+1}})$$

We say that $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_{i+1} \rangle$ is admissible for p if, in the forcing $P \restriction_{g_{i+1}}$, the sequence $\vec{\nu}_{i+1} = \langle \nu_{i+1}, \nu^1_{i+1}, \ldots, \nu^{m_{i+1}-1}_{i+1} \rangle$ is admissible for $q \restriction_{g_{i+1}}$ in the sense of definition 3.9, and if–

$$\left(q \restriction_{g_{i+1}}\right)^\frown \langle \vec{\nu}_{i+1} \rangle \Vdash \langle t_{i+1}{}^\frown \langle \nu_{i+1} \rangle, \underset{\sim}{A}^q_{g_{i+1}} \rangle \text{ extends } q\left(g_{i+1}\right), \text{ and } \langle F^j_{i+1} \colon j \leq m_{\alpha_{i+1}} \rangle$$
$$\text{is the system of measures } \langle U^j_{g_{i+1}} \colon j \leq m\left(U^{\times}_{g_{i+1}}\right) \rangle.$$

Assuming this holds, let–

$$p^\frown \langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_{i+1} \rangle = \left(\left(q \restriction_{g_{i+1}}\right)^\frown \langle \vec{\nu}_{i+1} \rangle\right)^\frown \langle t_{i+1}{}^\frown \langle \nu_{i+1} \rangle, \underset{\sim}{A}^q_{g_{i+1}} \rangle^\frown q \setminus \left(g_{i+1} + 1\right)$$

Given $i < \omega$ and a condition p which forces that–

$$\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i \rangle = \langle \vec{\mu}^*_0(\xi), \vec{\mu}^*_1(\xi), \ldots, \vec{\mu}^*_i(\xi) \rangle$$

We define, similarly to above, whenever a sequence

$$\langle \langle \nu_{i+1}, \nu^1_{i+1}, \ldots, \nu^{m_{i+1}-1}_{i+1} \rangle, \ldots\ldots, \langle \nu_k, \nu^1_k, \ldots, \nu^{m_k-1}_k \rangle \rangle$$

is admissible for p above $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i \rangle$. If this is the case, we can define similarly the condition $p^\frown \langle \langle \nu_{i+1}, \nu^1_{i+1}, \ldots, \nu^{m_{i+1}-1}_{i+1} \rangle, \ldots\ldots, \langle \nu_k, \nu^1_k, \ldots, \nu^{m_k-1}_k \rangle \rangle$.

Theorem 3.13 (Multivariable Fusion)**.** *Let $p \in P_\kappa$ be a condition and, for every sequence–*

$$\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle = \langle \langle \xi, \xi^1, \ldots, \xi^m \rangle, \langle \nu_1, \nu^1_1, \ldots, \nu^{m_1-1}_1 \rangle, \ldots, \langle \nu_k, \nu^1_k, \ldots, \nu^{m_k-1}_k \rangle \rangle$$

let $e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ be a P_{ν_k}-name for a subset of $P \setminus \nu_k$ which is $\leq^*$ dense open above any condition which forces that $\langle \vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi) \rangle = \langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle$. Then there exists $p^* \geq^* p$ and a set $X \in \mathcal{E}^{'}_0$, such that for every sequence of increasing sequences,

$$\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle = \langle \langle \xi, \xi^1, \ldots, \xi^m \rangle, \langle \nu_1, \nu^1_1, \ldots, \nu^{m_1-1}_1 \rangle, \ldots, \langle \nu_k, \nu^1_k, \ldots, \nu^{m_k-1}_k \rangle \rangle$$

which is admissible for p^*, and such that $\langle \xi, \xi^1, \ldots, \xi^m \rangle \in X$,

$$p^{*\frown} \langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \restriction_{\nu_k} \Vdash p^{*\frown} \langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \setminus \nu_k \in e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$$

Furthermore, there exists $p^* \in G$, for which–

$$\begin{aligned}\{\xi < \kappa \colon & \langle \vec{\xi}, \vec{\mu}_{\alpha_1}(\xi), \ldots, \vec{\mu}_{\alpha_k}(\xi) \rangle \text{ is admissible for } p^*,\\ & p^{*\frown} \langle \vec{\xi}, \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi) \rangle \in G \text{ and}\\ & p^{*\frown} \langle \vec{\xi}, \vec{\mu}^*_{\alpha_1}, \ldots, \vec{\mu}^*_{\alpha_k} \rangle \restriction_{\mu_{\alpha_k}} \Vdash p^{*\frown} \langle \vec{\xi}, \vec{\mu}^*_{\alpha_1}, \ldots, \vec{\mu}^*_{\alpha_k} \rangle \setminus \mu_{\alpha_k} \in e\left(\vec{\xi}, \vec{\mu}^*_{\alpha_1}, \ldots, \vec{\mu}^*_{\alpha_k}\right)\} \in W\end{aligned}$$

Proof. For every $1 \leq i \leq k$ and a sequence $\langle \vec{\xi}, \ldots, \vec{\nu}_i \rangle$, we define a set $e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\right)$, which is $\leq^*$ dense open above conditions which force that–

$$\langle d^{-1}\{\xi\}, \langle \mu_{\alpha_1}(\xi), \mu^{*1}_{\alpha_1}(\xi), \ldots, \mu^{*m_1-1}_{\alpha_1}(\xi)\rangle, \ldots, \langle \mu_{\alpha_i}(\xi), \mu^{*1}_{\alpha_i}(\xi), \ldots, \mu^{*m_i-1}_{\alpha_i}(\xi)\rangle\rangle =$$
$$\langle\langle \xi^1, \ldots, \xi^m\rangle, \langle \nu_1, \nu_1^1, \ldots, \nu_1^{m_1-1}\rangle, \ldots, \langle \nu_i, \nu_i^1, \ldots, \nu_i^{m_i-1}\rangle\rangle$$

as follows:

$$\begin{aligned} e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\right) = \{ & r \in P \setminus \nu_i \colon \text{ for every } \langle \vec{\nu}_{i+1}, \ldots, \vec{\nu}_k\rangle \text{ which is admissible for} \\ & r \text{ above } \langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\rangle, r^\frown\langle \vec{\nu}_{i+1}, \ldots, \vec{\nu}_k\rangle \restriction_{\nu_k} \Vdash \\ & r^\frown\langle \vec{\nu}_{i+1}, \ldots, \vec{\nu}_k\rangle \setminus \nu_k \in e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)\} \end{aligned}$$

Lemma 3.14. *If* $e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i, \vec{\nu}_{i+1}\right)$ *is $\leq^*$-dense open above conditions which force that–*

$$\langle \vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_{i+1}}(\xi)\rangle = \langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i, \vec{\nu}_{i+1}\rangle$$

then $e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\right)$ *is $\leq^*$-dense open above conditions which force that–*

$$\langle \vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_i}(\xi)\rangle = \langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\rangle$$

Proof. Fix $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\rangle$. Let $r \in P \setminus \nu_i$ be a condition which forces that–

$$\langle \vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_i}(\xi)\rangle = \langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\rangle$$

Denote for simplicity $m = m_{i+1}$ and–

$$g_{i+1} = g_{i+1}\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\right), \; t_{i+1} = f_{i+1}\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\right), \; F^j_{i+1} = F^j_{i+1}\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_i\right) \; (0 \leq j < m)$$

We apply theorem 3.10. For that, consider the forcing $P \restriction_{(\nu_i, g_{i+1})}$ and the sequence $F^0_{i+1} \lhd F^1_{i+1} \lhd \ldots \lhd F^{m-1}_{i+1}$ of measures on g_{i+1}. We describe a set $d(\nu_{i+1}, \nu^1_{i+1}, \ldots, \nu^{m-1}_{i+1}) \subseteq P \restriction_{(\nu_i, g_{i+1})} \setminus \nu_{i+1}$ which is $\leq^*$ dense open above conditions which force that $d^{-1}\{\nu_{i+1}\} = \langle \nu^1_{i+1}, \ldots, \nu^{m-1}_{i+1}\rangle$:

$$\begin{aligned} d(\nu_{i+1}, \nu^1_{i+1}, \ldots, \nu^{m-1}_{i+1}) = \{ & s \in P \restriction_{(\nu_{i+1}, g_{i+1})} \colon \text{ if } s \Vdash \langle t_{i+1}{}^\frown\langle \nu_{i+1}\rangle, \underset{\sim}{A}^r_{g_{i+1}}\rangle \geq r(g_{i+1}), \text{ then there} \\ & \text{exists a direct extension } q \geq^* \langle t_{i+1}{}^\frown\langle \nu_{i+1}\rangle, \underset{\sim}{A}^r_{g_{i+1}}\rangle^\frown r \setminus (g_{i+1}+1) \\ & \text{such that } s^\frown q \in e\left(\vec{\xi}, \ldots, \vec{\nu}_i, \vec{\nu}_{i+1}\right)\} \end{aligned}$$

By theorem 3.10, there exists $r^* \restriction_{g_{i+1}} \geq^* r \restriction_{g_{i+1}}$ and a set X which belongs to the product measure $\mathcal{E}_{i+1} = F^0_{i+1} \times F^1_{i+1} \times \ldots \times F^{m-1}_{i+1}$, such that for every increasing $\langle \nu_{i+1}, \nu^1_{i+1}, \ldots, \nu^{m-1}_{i+1}\rangle \in X$,

$$r^* \restriction_{\nu_{i+1}} \Vdash r^{*\frown}\langle \nu_{i+1}, \nu^1_{i+1}, \ldots, \nu^{m-1}_{i+1}\rangle \setminus \nu_{i+1} \in d\left(\nu_{i+1}, \nu^1_{i+1}, \ldots, \nu^{m-1}_{i+1}\right)$$

Let us define $r^* \setminus g_{i+1}$. Assume that we work in the generic extension for $P_{g_{i+1}}$, and $r^* \restriction_{g_{i+1}}$, which is already defined, belongs to it. For every $\nu_{i+1} \in A^r_{g_{i+1}}$ above $\max t_{i+1}$, we denote $d^{-1}(\nu_{i+1}) = \langle \nu^1_{i+1}, \ldots, \nu^{m-1}_{i+1}\rangle$. Let $q(\nu_{i+1}) \geq^* \langle t_{i+1}{}^\frown\langle \nu_{i+1}\rangle, A^r_{g_{i+1}}\rangle^\frown r \setminus g_{i+1}$ be a condition such that–

$$\left(r^{*\frown}\langle \nu_{i+1}, \ldots, \nu^{m-1}_{i+1}\rangle \setminus \nu_{i+1}\right)^\frown q(\nu_{i+1}) \in e\left(\vec{\xi}, \ldots, \vec{\nu}_{i+1}\right)$$

(and take $q(\nu_{i+1}) = r \setminus g_{i+1}$ if such q does not exist).

We can now define $r^*(g_{i+1})$. Generate from the set $X \in \mathcal{E}_{i+1}$ a corresponding set $Y \in W_{g_{i+1}}$. Just pick Y to be a set such that every increasing sequence from $Y \times \pi_{1,0}^{-1} Y \times \ldots \times \pi_{m-1,0}^{-1} Y$ belongs to X. Let–

$$r^*(g_{i+1}) = \langle \langle \rangle, A^r_{g_{i+1}} \cap (Y \cup \max t_{i+1}) \cap \left(\left(\triangle_{\nu_{i+1} < g_{i+1}} A^{q(\nu_{i+1})}_{g_{i+1}} \right) \cup \max t_{i+1} \right) \rangle$$

Finally, let us define $r^* \setminus (g_{i+1}+1)$ to be $q(\underset{\sim}{\nu}_{i+1}) \setminus (g_{i+1}+1)$, where here, $\underset{\sim}{\nu}_{i+1} = d(g_{i+1})$ can be read from the generic up to $g_{i+1}+1$. This concludes the definition of $r^* \in e\left(\vec{\xi}, \ldots, \vec{\nu}_i\right)$. $\square$

Inductively, it follows that for every increasing sequence $\vec{\xi} = \langle \xi, \xi^1, \ldots, \xi^m \rangle$, the set $e(\vec{\xi})$, defined similarly as above, is $\leq^*$-dense open above conditions which force that $d^{-1}\{\xi\} = \langle \xi^1, \ldots, \xi^m \rangle$. Apply theorem 3.10 one more time to obtain, from the condition p given in the formulation of the theorem, the required direct extension p^*. $\square$

3.4 Proof of Properties (B)-(E)

Lemma 3.15. *$\mu_\alpha = crit(k_\alpha)$ is measurable in M_α.*

Proof. Write $\mu_\alpha = j_\alpha(h)\left(j_{1,\alpha}\left([Id]_0' \right), j_{\alpha_1+1,\alpha}\left([Id]_{\alpha_1} \right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k} \right) \right)$. We can assume that for every $\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k$,

$$h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) > \nu_k$$

since $\mu_\alpha > \mu_{\alpha_k}$ (this can be done by modifying the value of $h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ to 0 whenever $h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \leq \nu_k$). We can also assume similarly that $h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ is a regular cardinal.

Let $f \in V[G]$ be a function such that $\mu_\alpha = [f]_W$. Let us assume, for contradiction, that for every $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle$, $h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ is non-measurable.

By changing f on a set outside of W, we can also assume that for every $\xi < \kappa$,

$$f(\xi) < h\left(\vec{\xi}, \vec{\mu}_{\alpha_1}(\xi), \ldots, \vec{\mu}_{\alpha_k}(\xi)\right) \tag{5}$$

Indeed, this can be done since–

$$[f]_W = \mu_\alpha < k_\alpha(\mu_\alpha) = \left[\xi \mapsto h\left(\vec{\xi}, \vec{\mu}_{\alpha_1}(\xi), \ldots, \vec{\mu}_{\alpha_k}(\xi)\right) \right]_W$$

Let $p \in G$ be a condition which forces that for every $\xi < \kappa$, equation 5 holds. From now on, work with conditions in $P = P_\kappa$ above p. We define, for every–

$$\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle = \langle \langle \xi, \xi^1, \ldots, \xi^m \rangle, \langle \nu_1, \nu_1^1, \ldots, \nu_1^{m_1-1} \rangle, \ldots, \langle \nu_k, \nu_k^1, \ldots, \nu_k^{m_k-1} \rangle \rangle$$

a set–

$$e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \subseteq P \setminus \nu_k$$

which is $\leq^*$ dense open above conditions which force that $\langle \vec{\mu}_{\alpha_1}(\xi), \ldots, \vec{\mu}_{\alpha_k}(\xi) \rangle = \langle \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle$:

$$e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) = \{ r \in P \setminus \nu_k \colon \exists \alpha < h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right),\ r \Vdash \underset{\sim}{f}(\xi) < \check{\alpha} \}$$

The $\leq^*$-density of $e\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$ essentially uses the fact that $h = h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$ is not measurable; employing this, $\underset{\sim}{f}(\xi)$ can be reduced to a P_h-name, and thus be evaluated by less then h many possibilities by lemma 1.8.

Let us apply now the Multivariable Fusion Lemma. There exists $p^* \in G$ and sets $X_0 \in U^0,\ldots,X_m \in U^m$ such that for every sequence of sequences $\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle$ which is admissible for p^*,

$$\left(p^{*\frown}\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle\right)\restriction_{\nu_k}\Vdash\left(p^{*\frown}\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle\right)\setminus\nu_k\in e\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$$

whenever $\vec{\xi}\in X_0\times\ldots X_m$ is increasing.

For every such $\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle$, let–

$$A\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)=\{\gamma<h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right):\exists q\geq\left(p^{*\frown}\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle\right)\restriction_{\nu_k},$$
$$q\Vdash\gamma=\underset{\sim}{\alpha}\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)\}$$

where $\underset{\sim}{\alpha}\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$ is the P_{ν_k}-name for the ordinal α which witnesses the fact that $\left(p^{*\frown}\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle\right)\setminus\nu_k\in e\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$. Then $A\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$ is a bounded subset of $h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$, since $\nu_k<h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$ and $\mathrm{GCH}_{\leq\kappa}$ holds in V.

For a set of ξ-s in W,

$$p^*\Vdash\underset{\sim}{f}(\xi)\in A\left(\vec{\xi},\vec{\mu}^*_{\alpha_1}(\xi),\ldots,\vec{\mu}^*_{\alpha_k}(\xi)\right)$$

In particular, in $M[H]$,

$$[f]_W\in j_W\left(\langle\vec{\xi},\vec{\nu}_0,\ldots,\vec{\nu}_k\rangle\mapsto A\left(\vec{\xi},\vec{\nu}_0,\ldots,\vec{\nu}_k\right)\right)\left(\vec{\kappa}^*,\vec{\mu}^*_{\alpha_1},\ldots,\vec{\mu}^*_{\alpha_k}\right)=$$
$$k_\alpha\left(j_\alpha\left(\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle\mapsto A\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)\right)\left(j_{1,\alpha}\left([Id]_0^{'}\right),j_{\alpha_1+1,\alpha}\left([Id]_{\alpha_1}\right),\ldots,j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right)\right)$$

But–

$$\left|j_\alpha\left(\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle\mapsto A\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)\right)\left(j_{1,\alpha}\left([Id]_0^{'}\right),j_{\alpha_1+1,\alpha}\left([Id]_{\alpha_1}\right),\ldots,j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right)\right|<$$
$$j_\alpha(h)\left(j_{1,\alpha}\left([Id]_0^{'}\right),j_{\alpha_1+1,\alpha}\left([Id]_{\alpha_1}\right),\ldots,j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right)=\mu_\alpha$$

and thus $\mu_\alpha=[f]_W\in\mathrm{Im}(k_\alpha)$, a contradiction. $\square$

Corollary 3.16. *μ_α is the least measurable μ in M_α such that $\mu\geq\sup\{\mu_{\alpha'}:\alpha'<\alpha\}$ and $(cf(\mu))^V>\kappa$.*

Proof. Assume for contradiction that there exists a measurable λ in M_α, such that $\sup\{\mu_{\alpha'}:\alpha'<\alpha\}\leq\lambda<\mu_\alpha$. Let us argue that $(\mathrm{cf}(\lambda))^V\leq\kappa$.

First, $\lambda=k_\alpha(\lambda)$, since $\mathrm{crit}(k_\alpha)=\mu_\alpha$, and thus λ is measurable in M. Note that $\lambda<\mu_\alpha<\kappa^*=j_\alpha(\kappa)$, so $\lambda<j_W(\kappa)$ and thus λ has cofinality ω in $M[H]$. Thus, $(\mathrm{cf}(\lambda))^{V[G]}=\omega$, and, in particular, $(\mathrm{cf}(\lambda))^V<\kappa$. $\square$

Lemma 3.17. *μ_α appears as an element in the Prikry sequence of $k_\alpha(\mu_\alpha)$.*

Proof. In $M[H]$, denote by t^* the initial segment of the Prikry sequence of $k_\alpha(\mu_\alpha)$ which consists of all the ordinals below μ_α. Denote by n^* the length of t^*. Let $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \mapsto t^*\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ be a function in V such that–

$$t^* = j_\alpha\left(\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \mapsto t^*\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)\right)\left(j_{1,\alpha}\left([Id]^{'}_\alpha\right), j_{\alpha_1+1,\alpha}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right)$$

(this can be done by modifying the nice sequence $\langle \vec{\alpha}_1, \ldots, \vec{\alpha}_k \rangle$, if necessary, so that t^* can be represented by it). We can assume that for every $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle$, $t^*\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ is a sequence of length n^*. Since $k_\alpha(t^*) = t^*$,

$$\left[\xi \mapsto t^*\left(\vec{\xi}, \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)\right]_W = t^*$$

In $V[G]$, denote, for every $\xi < \kappa$,

$$\mu_\alpha(\xi) = \text{ the } (n^*+1)\text{-th element in the Prikry sequence of } h\left(\vec{\xi}, \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$$

Clearly $[\xi \mapsto \mu_\alpha(\xi)]_W \geq \mu_\alpha$. We argue that equality holds. We will prove that for every $\eta < [\xi \mapsto \mu_\alpha(\xi)]_W$, $\eta < \mu_\alpha$. Assume that such η is given, and let $f \in V[G]$ be a function such that $[f]_W = \eta$. Then we can assume that for every $\xi < \kappa$,

$$f(\xi) < \mu_\alpha(\xi)$$

and let $p \in G$ be a condition which forces this.

Let us apply now the Multivariable Fusion Lemma. For every $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle$, consider the set–

$$\begin{aligned} e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) = \{r \in P \setminus \nu_k \colon \exists \alpha < h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right),\ r \Vdash \text{ if } t^*\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \\ \text{is an initial segment of the Prikry sequence of } h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right), \\ \text{then } \underset{\sim}{f}(\xi) < \alpha\} \end{aligned}$$

then $e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ is $\leq^*$ dense open above conditions which force that $d^{-1}\{\xi\} = \langle \xi^1, \ldots, \xi^m \rangle$ and $\langle \vec{\nu}_0, \ldots, \vec{\nu}_k \rangle = \langle \vec{\mu}_{\alpha_0}(\xi), \ldots, \vec{\mu}_{\alpha_k}(\xi) \rangle$. This follows in several steps: First, use the $\leq^*$-closure to reduce $\underset{\sim}{f}(\xi)$ to a P_{h+1}-name, where $h = h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$. This can be done by taking a direct extension of a given condition in the forcing $P \setminus (h+1)$. Second, reduce $\underset{\sim}{f}(\xi)$ to a P_h name, by applying on the Prikry forcing at coordinate h the following fact: If $t^* = t^*\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ is an initial segment of the Prikry sequence of h, and a given an ordinal is forced to be below the successor of t^* in this sequence, then its value can be decided by taking a direct extension. Finally, apply lemma 1.10 and direct extend in the forcing $P_{(\nu_k, h)}$, to bound the value of $\underset{\sim}{f}(\xi)$ by an ordinal below h.

Thus, there exists $p^* \in G$, such that for every $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle$ which is admissible for p^*,

$$p^{*\frown}\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \restriction_{\nu_k} \Vdash p^{*\frown}\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \setminus \nu_k \in e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$$

in particular, $p^{*\frown}\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \restriction_{\nu_k}$ forces that there exists $\alpha < h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ such that–

$$p^{*\frown}\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \setminus \nu_k \Vdash \underset{\sim}{f}(\xi) < \check{\alpha} \tag{6}$$

Let–

$$A\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)=\{\gamma\colon \exists q\geq p^{*\frown}\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle\restriction_{\nu_k},\ q\Vdash \underset{\sim}{\alpha}=\gamma\}$$

(where, as in the previous lemma, $\underset{\sim}{\alpha}$ is a P_{ν_k}-name for the ordinal α in equation 6).

Then $p^*\Vdash \underset{\sim}{f}(\xi)\in A\left(\vec{\xi},\vec{\mu}^*_{\alpha_1}(\xi),\ldots,\vec{\mu}^*_{\alpha_k}(\xi)\right)$, and $A\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$ is a bounded subset of $h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$. Arguing as lemma 3.15, it follows that $\eta=[f]_W\in\operatorname{Im}(k_\alpha)\cap k_\alpha(\mu_\alpha)=\mu_\alpha$, as desired. $\square$

Lemma 3.18. *Assume that for every measurable $\xi<\kappa$, $\underset{\sim}{U}_\xi$ is a P_ξ-name for a measure on ξ which belongs to V. Let $\underset{\sim}{U}=j_\alpha(\xi\mapsto\underset{\sim}{U}_\xi)(\mu_\alpha)$. Then there exists a set $\mathcal{F}\in M_\alpha$ of measures on μ_α in M_α, with $|\mathcal{F}|<\mu_\alpha$, such that, for some $p\in G$,*

$$(j_W(p))^\frown\langle\vec{\mu}^*_{\alpha_0},\ldots,\vec{\mu}^*_{\alpha_k}\rangle\Vdash k_\alpha(\underset{\sim}{U})\in k''_\alpha\mathcal{F}$$

In particular, there exists a measure $F\in\mathcal{F}$ such that $k_\alpha(F)=(k_\alpha(\underset{\sim}{U}))_H$.

Proof. For every $\xi<\kappa$, fix an enumeration s_ξ of all the normal measures on ξ in V. Apply Multivariable Fusion. Define for every $\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle$ the set–

$$\begin{aligned}e\left(\vec{\xi},\vec{\nu}_0,\ldots,\vec{\nu}_k\right)=&\{r\in P\setminus\nu_k\colon \text{there exists a set of ordinals } A \text{ of cardinality}\\&\text{strictly smaller than } h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right),\text{ such that}\\&r\restriction_{h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)}\Vdash \underset{\sim}{U}_{h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)}\in s''_{h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)}A\}\end{aligned}$$

As before, there exists $p^*\geq^* p$ in G, such that for every $\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle$ which is admissible for p^*,

$$p^{*\frown}\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle\restriction_{\nu_k}\Vdash p^{*\frown}\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle\setminus\nu_k\in e\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$$

Let $A\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$ be the set of ordinals, forced by some extension of $p^{*\frown}\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle\restriction_{\nu_k}$, to be an element of the set $\underset{\sim}{A}$ above. Then $A\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$ is a set of ordinals of cardinality strictly below $h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)$.

In $V[G]$,

$$\{\xi<\kappa\colon p^{*\frown}\langle\vec{\xi},\vec{\mu}^*_{\alpha_1}(\xi),\ldots,\vec{\mu}^*_{\alpha_k}(\xi)\rangle\Vdash U_{h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)}\in s''_{h\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)}A^*\left(\vec{\xi},\vec{\mu}^*_{\alpha_1}(\xi),\ldots,\mu^*_{\alpha_k}(\xi)\right)\}\in W$$

Now, in M_α, denote–

$$\mathcal{F}=(j_\alpha(s)_{\mu_\alpha})''\,j_\alpha\left(\langle\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\rangle\mapsto A^*\left(\vec{\xi},\vec{\nu}_1,\ldots,\vec{\nu}_k\right)\right)\left(j_{1,\alpha}\left([Id]_0^{'}\right),j_{\alpha_1+1,\alpha}\left([Id]_1\right),\ldots,j_{\alpha_k+1,\alpha}\left([Id]_k\right)\right)$$

Then $|\mathcal{F}|<\mu_\alpha$, and, in $M[H]$,

$$(j_W(p^*))^\frown\langle\vec{\mu}^*_{\alpha_0},\ldots,\vec{\mu}^*_{\alpha_k}\rangle\Vdash k_\alpha(\underset{\sim}{\mathcal{U}})\in k''_\alpha\mathcal{F}$$

$\square$

We now apply the above lemma on specific names for measures $\underset{\sim}{U}_\xi$. For every measurable $\xi < \kappa$, let $W_\xi = U^\times_\xi$ be the measure used to singularize ξ at stage ξ in the iteration $P = P_\kappa$. Each such $W^\times_\xi$ can be assigned to a sequence of Rudin-Keisler equivalent measures on ξ,

$$W^0_\xi, \ldots, W^{m_\xi}_\xi = W_\xi$$

as defined in section 2. Denote $U^j_\xi = W^j_\xi \cap V \in V$ for every $0 \le j \le m_\xi$. Then–

$$U^0_\xi \lhd U^1_\xi \lhd \ldots \lhd U^{m_\xi}_\xi$$

Corollary 3.19. *For every $1 \le j < m_\alpha$, there exists a measure $U^j_{\mu_\alpha} \in M_\alpha$ on μ_α, such that–*

$$k_\alpha\left(U^j_{\mu_\alpha}\right) = j_W\left(\xi \mapsto U^j_\xi\right)\left(k_\alpha\left(\mu_\alpha\right)\right) = \left[\xi \mapsto U^j_{h\left(\vec{\mu}^*_{\alpha_0}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)}\right]_W$$

We consider the above corollary as the definition of the measures $U^j_{\mu_\alpha}$ for every $1 \le j < m_\alpha$. Note that, by elementarity, $U^0_{\mu_\alpha} \lhd U^1_{\mu_\alpha} \lhd \ldots \lhd U^{m_\alpha - 1}_{\mu_\alpha}$. Let $\mathcal{E}_\alpha$ be the measure on $[\kappa]^{m_\alpha}$ which corresponds to the iterated ultrapower with $U^{m_\alpha-1}_{\mu_\alpha} \rhd \ldots \rhd U^0_{\mu_\alpha}$ in decreasing order. We argue that $\mathcal{E}_\alpha$ is derived from $k_\alpha \colon M_\alpha \to M$.

Lemma 3.20. *$U^0_{\mu_\alpha} = \{X \subseteq \mu_\alpha \colon \mu_\alpha \in k_\alpha(X)\} \cap M_\alpha$. Furthermore, if $m_\alpha > 1$, then for every $1 \le j \le m_\alpha - 1$,*

$$U^j_{\mu_\alpha} = \{X \subseteq \mu_\alpha \colon \mu^{*j}_\alpha \in k_\alpha(X)\} \cap M_\alpha$$

Remark 3.21.

1. *Let us note that for every j as above, $\mu_\alpha < \mu^j_\alpha < k_\alpha\left(\mu_\alpha\right)$, so $U^{*j}_{\mu_\alpha}$ is an ultrafilter which concentrates on μ_α.*
2. *We deliberately did not define, in corollary 3.19, the measure $U^{m_\alpha}_{\mu_\alpha}$ - it is not derived from k_α and does not participate in $j_W \restriction_V$. The exception is $\alpha = 0$ where $U^m = U$ is the first step in the iteration.*

Proof. We first provide the proof for $U^0_{\mu_\alpha}$. Assume that $X \in M_\alpha$ and $\mu_\alpha \in k_\alpha(X)$. Write–

$$X = j_\alpha\left(\langle \vec{\xi}, \vec{\nu}_0, \ldots, \vec{\nu}_k\rangle \mapsto X\left(\vec{\xi}, \vec{\nu}_0, \ldots, \vec{\nu}_k\right)\right)\left(j_{1,\alpha}\left(\vec{\kappa}\right), j_{\alpha_0,\alpha}\left(\vec{\mu}_{\alpha_0}\right), \ldots, j_{\alpha_k,\alpha}\left(\vec{\mu}_{\alpha_k}\right)\right)$$

where, without loss of generality, the nice sequence $\langle \alpha_0, \ldots, \alpha_k\rangle$ can be used to represent μ_α in M_α, in the usual sense that for a function $h \in V$,

$$\mu_\alpha = j_\alpha(h)\left(j_{1,\alpha}\left(\vec{\kappa}\right), j_{\alpha_0,\alpha}\left(\vec{\mu}_{\alpha_0}\right), \ldots, j_{\alpha_k,\alpha}\left(\vec{\mu}_{\alpha_k}\right)\right)$$

Apply Multivariable Fusion. For every $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle$, let–

$$\begin{aligned} e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) = \{r \in P \setminus \nu_k \colon\ & r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \parallel X\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \in U^\times_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)}, \\ & \text{if it decides positively, then } r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \Vdash \underset{\sim}{A}^r_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \subseteq \\ & X\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right);\ \text{else, } r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \Vdash \underset{\sim}{A}^r_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \text{ is disjoint} \\ & \text{from } X\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right).\ \text{Moreover, } r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \parallel \mathrm{lh}\left(t^r_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)}\right) > n^*, \\ & \text{and if it decides positively, then there exists a bounded subset} \\ & A\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \subseteq h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \text{ for which } r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \Vdash \text{ the } n^*\text{-th} \\ & \text{element of } t^r_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \text{ belongs to } A\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)\} \end{aligned}$$

Applying the same tools above, there exists a condition $p^* \in G$ and a bounded subset $A^*\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$, such that–

$$\begin{aligned}\{\xi < \kappa \colon p^* \restriction_{h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)} \Vdash \text{ if } \mathrm{lh}\left(t^r_{h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)}\right) \geq n^* \text{ then the}\\ &n^*\text{-th element in the Prikry sequence of}\\ &h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right) \text{ belongs to } A^*\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)\} \in W\end{aligned}$$

Since $A^*\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$ is bounded in $h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$, it follows that–

$$\{\xi < \kappa \colon p^* \restriction_{h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)} \Vdash \mathrm{lh}\left(t^r_{h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)}\right) < n^*\} \in W$$

By the choice of p^*, it follows that for a set of ξ-s in W,

$$p^* \restriction_{h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)} \Vert\, X\left(\xi, \mu_{\alpha_0}(\xi), \ldots, \mu_{\alpha_k}(\xi)\right) \in U^{\times}\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$$

we argue that for a set of ξ-s in W, the decision is positive. Indeed, otherwise, it holds in $M[H]$ that–

$$\mu_\alpha = [\xi \mapsto \mu_\alpha(\xi)]_W \in \left[\xi \mapsto h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right) \setminus X\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)\right]_W = k_\alpha(\mu_\alpha \setminus X)$$

contradicting the choice of X. Thus, for a set of ξ-s in W,

$$p^* \restriction_{h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)} \Vdash X\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right) \in U^{\times}\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$$

recall that $U^0 = U^{\times} \cap V$; hence–

$$p^* \restriction_{h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)} \Vdash X\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right) \in U^{0}\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$$

and thus, in $M[H]$, $k_\alpha(X) \in k_\alpha\left(U^0_{\mu_\alpha}\right)$. In particular, in M_α, $X \in U^0_{\mu_\alpha}$.

We now proceed to the proof for $U^j_{\mu_\alpha}$ for every $1 \leq j \leq m_\alpha - 1$. Assume that $\mu^{*j}_\alpha \in k_\alpha(X)$, and recall that μ^{*j}_α is the j-th element in $d^{-1}(\mu_\alpha)$. We repeat the same argument above. First, define–

$$\begin{aligned} e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) = \{r \in P \setminus \nu_k \colon\ & r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \Vert\, X\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \in U^j_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)},\\ & \text{if it decides positively, then } r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \Vdash \underset{\sim}{A}^r_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \subseteq\\ & d''X\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right);\ \text{else, } r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \Vdash \pi^{-1}_{j,0}{}'' \underset{\sim}{A}^r_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \text{ is disjoint}\\ & \text{from } X\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right). \text{ Moreover, } r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \Vert\, \mathrm{lh}\left(t^r_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)}\right) > n^*,\\ & \text{and if it decides positively, then there exists a bounded subset}\\ & A\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \subseteq h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) \text{ for which } r \restriction_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \Vdash \text{ the } n^*\text{-th}\\ & \text{element of } t^r_{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)} \text{ belongs to } A\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)\}\end{aligned}$$

Now we argue as before, and claim that–

$$p^* \restriction_{h\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)} \Vdash X\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right) \in U^{j}\left(\vec{\mu}^*_0(\xi), \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$$

indeed, otherwise, there exists a set of ξ-s in W for which–

$$\mu_\alpha^j(\xi) = \pi_{j,0}^{-1}\left(\mu_\alpha(\xi)\right) \notin X\left(\vec{\mu}_0^*(\xi), \vec{\mu}_{\alpha_1}^*(\xi), \dots, \vec{\mu}_{\alpha_k}^*(\xi)\right)$$

contradicting the fact that $\mu_\alpha^j \in k_\alpha(X)$. It follows that, in $M[H]$, $k_\alpha(X) \in k_\alpha\left(U^j_{\mu_\alpha}\right)$, and so $X \in U^j_{\mu_\alpha}$. $\square$

Corollary 3.22. $\mathcal{E}_\alpha = \{X \subseteq [\mu_\alpha]^{m_\alpha} : \langle \mu_\alpha, \mu_\alpha^{*1}, \dots, \mu_\alpha^{*m_\alpha - 1}\rangle \in k_\alpha(X)\} \cap M_\alpha$.

Proof. It suffices to prove that–

$$\mathcal{E}_\alpha \subseteq \{X \subseteq [\mu_\alpha]^{m_\alpha} : \langle \mu_\alpha, \mu_\alpha^{*1}, \dots, \mu_\alpha^{*m_\alpha - 1}\rangle \in k_\alpha(X)\} \cap M_\alpha$$

Start from $X \in \mathcal{E}_\alpha$. Then there are sets $X_0 \in U^0_{\mu_\alpha}, \dots, X_{m_\alpha - 1} \in U^{m_\alpha-1}_{\mu_\alpha}$ such that the set of increasing sequences in $X_0 \times \ldots \times X_{m-1}$ is contained in X. Thus every increasing sequence in $k_\alpha(X_0) \times \ldots \times k_\alpha(X_{m-1})$ belongs to $k_\alpha(X)$, and by the previous lemma, $\langle \mu_\alpha, \dots, \mu_\alpha^{*m_\alpha-1}\rangle \in k_\alpha(X)$, as desired. $\square$

This concludes the proof of properties $(A)-(F)$ from the beginning of the section. We now focus on the proof of theorem 0.2.

Recall that $\kappa^* = j_U(\kappa)$. Note that $\kappa^* = j_{\kappa^*}(\kappa)$, since κ is mapped to κ^* after the first step in the iteration, and every step after it is taken on a measurable μ_α^j below κ^*. Moreover, $\sup\{\mu_\alpha \colon \alpha < \kappa^*\} = \kappa^*$ and thus $\operatorname{crit}(k_{\kappa^*}) \geq \kappa^*$. Let us use the above properties and argue that the induction halts after κ^*-many steps.

Proof of Theorem 0.2. Since $j_W \restriction_V = j_{\kappa^*} \circ k_{\kappa^*}$, it suffices to prove that $k_{\kappa^*} \colon M_{\kappa^*} \to M$ is the identity function. In particular, it will follow that $j_W \restriction_V = j_{\kappa^*}$, $M = M_{\kappa^*}$ and $j_W(\kappa) = j_{\kappa^*}(\kappa) = j_U(\kappa)$.

We argue that for every ordinal η, $\eta \in \operatorname{Im}(k_{\kappa^*})$. Fix such η and let $g \in V[G]$ be a function such that $[g]_{W^*} = \eta$. Let $\underset{\sim}{g}$ be a $P = P_\kappa$-name for it. For every $\xi < \kappa$, let–

$$e(\xi) = \{r \in P \setminus \xi \colon \text{for some } A \subseteq \kappa \text{ with } |A| < \kappa,\ r \Vdash \underset{\sim}{g}(\xi) \in A\}$$

$e(\xi)$ is $\leq^*$-dense open by lemma 1.8. By Fusion, there exists $p \in G$ such that for every $\xi < \kappa$,

$$p \restriction_\xi \Vdash \text{for some } A \subseteq \kappa \text{ with } |A| < \kappa,\ (p \setminus \xi)^{-\xi} \Vdash \underset{\sim}{g}(\xi) \in A$$

Fix $\xi < \kappa$ and let $\underset{\sim}{A}$ be a P_ξ-name for the above subset $A \subseteq \kappa$. Let–

$$A(\xi) = \{\gamma < \kappa \colon \exists q \geq p \restriction_\xi,\ q \Vdash \gamma \in \underset{\sim}{A}\}$$

Then for every $\xi < \kappa$, $|A(\xi)| < \kappa$, and $p^{-\xi} \Vdash \underset{\sim}{g}(\xi) \in A(\xi)$. Recall that for every $p \in G$, $(j_W(p))^{-[Id]_{W^*}} \in H$. Thus, in $M[H]$,

$$[g]_{W^*} \in j_{W^*}(\xi \mapsto A(\xi))\left([Id]_{W^*}\right) = k_{\kappa^*}\left(\ j_{\kappa^*}(\xi \mapsto A(\xi))\left(j_{1,\kappa^*}(\kappa)\right)\ \right)$$

but $|j_{\kappa^*}(\xi \mapsto A(\xi))(j_{1,\kappa^*}(\kappa))| < j_{\kappa^*}(\kappa) = \kappa^* \leq \operatorname{crit}(k_{\kappa^*})$, and thus $[g]_{W^*} \in \operatorname{Im}(k_{\kappa^*})$. $\square$

Lemma 3.23. *Fix $\alpha < \kappa^*$ and denote $m = m_\alpha$. Let $0 < j \leq m$. Then μ_α^{*j} is measurable in M, and its Prikry sequence in $M[H]$ is the sequence of critical points obtained by iterating the measure $U^{j-1}_{\mu_\alpha}$ over M_α.*

Proof. First, by lemma 3.17, μ_α appears in the Prikry sequence of μ^{*j}_α. Let λ be the element after it in this Prikry sequence, and let us argue that $\lambda = j_{U^{j-1}_{\mu_\alpha}}(\mu_\alpha) = \mu^{j-1}_\alpha$. Since $j_{U^{j-1}_{\mu_\alpha}}(\mu_\alpha)$ is measurable in $M_{\alpha+1}$ and has cofinality above κ in V, there exists $\beta > \alpha$ such that $\mu_\beta = \mu^j_\alpha$; Also, $k_\beta(\mu_\beta) = k_\beta\left(j_{\alpha+1,\beta}\left(\mu^j_\alpha\right)\right) = k_{\alpha+1}\left(\mu^j_\alpha\right) = \mu^{*j}_\alpha$, and thus $\mu_\beta = \mu^j_\alpha$ appears as an element in the Prikry sequence of μ^{*j}_α. Thus, $\lambda \leq \mu^j_\alpha$, and it suffices to prove that $\lambda = \mu^j_\alpha$. Assume for contradiction that $\lambda < \mu^{j-1}_\alpha = j_{U^{j-1}_{\mu_\alpha}}(\mu_\alpha)$.

In M_α write–

$$\mu_\alpha = j_\alpha(h)\left(j_{1,\alpha}\left([Id]^{'}_0\right), j_{\alpha_1+1,\alpha}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right)\right)$$

and–

$$\lambda = j_{\alpha+1}(g)\left(j_{1,\alpha}\left([Id]^{'}_0\right), j_{\alpha_1+1,\alpha}\left([Id]_{\alpha_1}\right), \ldots, j_{\alpha_k+1,\alpha}\left([Id]_{\alpha_k}\right), [Id]_\alpha\right)$$

for some functions f, g in V. Recall that $[Id]_\alpha = \langle \mu_\alpha, \ldots, \mu^{j-1}_\alpha, \ldots, \mu^{m_\alpha-1}_\alpha\rangle$, so we can assume that for every $\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k, \vec{\nu} = \langle \nu_0, \ldots, \nu_{m-1}\rangle$,

$$g\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k, \langle \nu^0, \ldots, \nu^{j-1}, \ldots, \nu^{m-1}\rangle\right) < \min\{h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right), \nu^{j-1}\}$$

Let t^* be the initial segment of the Prikry sequence of $k_\alpha(\mu_\alpha)$ which consists of all the ordinals below μ_α. Fix a function $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle \mapsto t^*\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ which represents t^* in M_α (as in lemma 3.17).

For simplicity, we adopt the following notation below: whenever $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle$ are fixed, let $h = h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$. Also, for every $\nu < h$, we denote $d^{-1}(\nu) = \langle \nu^1, \ldots, \nu^{m-1}\rangle$ (whenever $m \neq 1$). We also denote $\nu^0 = \nu$ and $\vec{\nu} = \langle \nu^0, \ldots, \nu^{m-1}\rangle$. Let $C = C\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)$ be the club of closure points of $\nu_0 \mapsto g\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k, \vec{\nu}\right)$ (this is a club in h. We remark that it is necessary in the proof below only in the case where $j = 1$).

We now apply the Multivariable Fusion lemma. Fix $\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\rangle$, and let–

$$\begin{aligned} e\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right) = \{ & r \in P \setminus \nu_k \colon \text{for every } \nu \in \underset{\sim}{A}^r_h, \\ & A^r_h \setminus \nu \subseteq \left(h \setminus \nu^{j-1}\right) \cap C\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right)\} \end{aligned}$$

First let us consider the case where $j > 1$. There exists $p \in G$ such that for a set of $\xi < \kappa$ in W, the condition $p^\frown\langle \vec{\xi}, \vec{\mu}^*_{\alpha_1}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\rangle$ forces that the element which appears after $\mu_\alpha(\xi)$ in the Prikry sequence of $h\left(\vec{\xi}, \vec{\mu}^*_{\alpha_0}(\xi), \ldots, \vec{\mu}^*_{\alpha_k}(\xi)\right)$ is strictly greater then $\mu^{*j-1}(\xi)$. Thus, in $M[H]$,

$$\lambda > \mu^{*j-1} > j_W(g)\left(\vec{\kappa}, \vec{\mu}^*_{\alpha_0}, \ldots, \vec{\mu}^*_{\alpha_k}, \vec{\mu}^*_\alpha\right) \geq \lambda$$

which is a contradiction.

If $j = 1$, we use the club C defined above: Since $\mu_\alpha < \lambda$, it follows that $\lambda > j_W(g)\left(\vec{\kappa}, \vec{\mu}^*_{\alpha_1}, \ldots, \vec{\mu}^*_{\alpha_k}, \vec{\mu}^*_\alpha\right)$ which is again a contradiction as above. $\square$

Corollary 3.24. *In $M[H]$, recall the sequence–*

$$d^{-1}\{\kappa\} = \langle \mu^{*1}_0, \ldots, \mu^{*m}_0\rangle = \langle [Id]_{W^1}, \ldots, [Id]_{W^m}\rangle$$

For every $1 \leq j \leq m$, the cardinal $\mu^{*j} = [Id]_{W^j}$ is measurable in M, and its Prikry sequence in $M[H]$ is the sequence of critical points in the iterated ultrapower, ω-many times, taken with the measure $U^{j-1} = W^{j-1} \cap V \in V$.

Finally, let us prove Theorem 0.3 and provide a sufficient condition for definability of $j_W \restriction_V$. Denote–

$$\vec{\mathcal{U}} = \langle U_\xi \colon \xi < \kappa \rangle$$

Recall that for every $\xi \in \Delta$, U_ξ is the measure on ξ in V such that $W_\xi = U^{\times}_\xi$. We will argue below that if $\vec{\mathcal{U}} \in V$, then $j_W \restriction_V$ is a definable class of V.

Remark 3.25. *The sequence $\vec{\mathcal{U}}$ might be external to V, even though every measure in it belongs to V. For instance, we may consider iterations where the measure used at stage $\alpha \in \Delta$ codes generic information about Prikry sequences below α.*

More specifically, Let $\eta = \min \Delta$ be the first measurable, and assume that there are unboundedly many measurables $\zeta < \kappa$ which carry at least η measures of Mitchell order 0 (see section 5 in [6] for a detailed construction of a model satisfying the above assumptions). Let $\langle \zeta_\xi \colon \xi < \kappa \rangle$ be an enumeration of the set of such measurables, and let $\langle U_{\zeta_\xi, \alpha} \colon \alpha < \eta \rangle$ be an enumeration of η-many measures of order 0 on each ζ_ξ.

Denote by $\langle \eta_n \colon n < \omega \rangle$ the Prikry sequence added to η in the iteration. For each $\xi < \kappa$, write $\xi = \xi' + n$ for ξ' limit and $n < \omega$. In the Prikry forcing at stage ζ_ξ in the iteration, use the measure $U^{\times}_{\zeta_\xi, \eta_n} = U^{*}_{\zeta_\xi, \eta_n}$. For every other measurable $\zeta \in \Delta$, use the extension of an arbitrary measure of order 0. Note that the forcing $P = P_\kappa$ generated this way uses only measures of order 0, so the iterated ultrapower is taken with measures $\mathcal{E}_\alpha = U^0_{\mu_\alpha}$, which are a single measure, and not a product of several measures on μ_α, for every $\alpha < \kappa^*$.

Let G be a generic set for the above iteration over V. Clearly $\vec{\mathcal{U}} \notin V$, as it codes the Prikry sequence $\langle \eta_n \colon n < \omega \rangle$ added to η.

Let us argue also that $j_W \restriction_V$ is not a definable class of V. As usual, let $j_W \colon V[G] \to M[H]$ be the ultrapower embedding. Let $\langle \lambda_n \colon n < \omega \rangle$ be an enumeration of the first ω-many measurables of $M_U = M_0$ which carry η-many measures. Then, by the analysis in this section, each λ_n carries a measure $U^{M_U}_{\lambda_n, \eta_n}$, which is iterated ω-many times in order to produce a Prikry sequence for a measurable $\lambda^*_n = j_{1,\kappa^*}(\lambda_n)$ of M. So λ_n remains measurable in M_{λ_n} (as all the steps in the iteration $\langle M_\alpha \colon \alpha < \lambda_n \rangle$ are ultrapowers on measurables below λ_n), and–

$$j_{\lambda_n+1} = j_{U^{M_{\lambda_n}}_{\lambda_n,\eta_n}} \circ j_{\lambda_n}$$

Here, $U^{M_{\lambda_n}}_{\lambda_n,\eta_n}$ is the η_n-th measure in the enumeration $j_{\lambda_n}(\zeta \mapsto \langle U_{\zeta,\alpha} \colon \alpha < \eta \rangle)(\lambda_n)$ of η-many normal measures of order 0 on λ_n in M_{λ_n}. Note that–

$$j_{1,\lambda_n}\left(U^{M_U}_{\lambda_n,\eta_n}\right) = U^{M_{\lambda_n}}_{\lambda_n,\eta_n}$$

Now, Assume for contradiction that $j_W \restriction_V$ is definable in V. Then the embedding $k \colon M_U \to M$,

$$k([f]_U) = [f]_{W^*} = j_W \restriction_V (f)([Id]_{W^*})$$

is definable in V as well. Fix a set $X \subseteq \lambda_n$ in M_U. Then–

$$\begin{aligned} X \in U^{M_U}_{\lambda_n,\eta_n} &\iff \lambda_n \in j_{U^{M_U}_{\lambda_n,\eta_n}}(X) \\ &\iff j_{1,\lambda_n}(\lambda_n) \in j_{1,\lambda_n}\left(j_{U^{M_U}_{\lambda_n,\eta_n}}(X)\right) \\ &\iff \lambda_n \in j_{1,\lambda_n+1}(X) \\ &\iff \lambda^*_n \in k(X) \end{aligned}$$

and thus $U^{M_U}_{\lambda_n,\eta_n}$, and in particular η_n, can be read from k. Thus, if $j_W \restriction_V$ is definable in V, then so is the sequence $\langle \eta_n : n < \omega \rangle$, which is a contradiction.

Finally. let us remark that it is possible that $j_W \restriction_V$ is definable in V, even though $\vec{\mathcal{U}} \notin V$. As above, let $\langle \eta_n : n < \omega \rangle$ be the Prikry sequence added to the first measurable η. On the first ω-many measurables $\langle \zeta_n : n < \omega \rangle$, choose the measures $\langle W_{\zeta_n} : n < \omega \rangle$ as above, namely $W_{\zeta_n} = U^*_{\zeta_n,\eta_n}$. For every other ζ_ξ ($\omega \leq \xi < \kappa$), use the measure derived from the least normal measure of order 0 on ζ_ξ in V, with respect to a prescribed well order $\mathcal{W}$ of V_κ. In this case $\mathcal{U} \notin V$, but $j_W \restriction_V$ is definable: For every $\alpha \leq \kappa^*$, the normal measure U_{μ_α} is chosen least, among normal measures of order 0 on μ_α, with respect to the well order $j_\alpha(\mathcal{W})$ (the use of the generic Prikry sequence added to η is done boundedly below κ, and thus does not influence the value of U_{μ_α}).

Proof of Theorem 0.3. We begin by proving the following claim:

Claim 3.26. *Assume that $\vec{\mathcal{U}} \in V$. Then $\langle\langle U^0_\xi, \ldots, U^{m_\xi - 1}_\xi, U^{m_\xi}_\xi = U_\xi \rangle : \xi \in \Delta \rangle \in V$ as well.*

Proof. We prove by induction on $\alpha \in \Delta$ that $\langle\langle U^0_\xi, \ldots, U^{m_\xi - 1}_\xi, U^{m_\xi}_\xi \rangle : \xi \in \Delta \cap \alpha \rangle$ is definable over V from U_α and $\langle U_\xi : \xi < \alpha \rangle$.

Fix $\alpha \in \Delta$. If U_α does not concentrate on $\Delta \cap \alpha$, then $m_\alpha = 0$ and $U^{m_\alpha}_\alpha = U_\alpha$. Assume otherwise. Denote $m = j_{U_\alpha}(\langle m_\xi : \xi \in \Delta \cap \alpha \rangle)(\alpha)$. We argue that $m_\alpha = m$. Consider the generic extension $V[G_\alpha]$ up to coordinate α. U_α concentrates on elements $\xi \in \Delta$ for which $m = m_\xi$. Each such $\xi \in \Delta$ satisfies that ξ is the m-th element in $d^{-1}\{d(\xi)\}$. Thus, in $\mathrm{Ult}(V[G_\alpha], U^\times_\alpha)$, $d^{-1}\{\kappa\} = m$ and thus $m(U^\times_\alpha) = m$. Thus indeed $m_\alpha = m$.

By the analysis done in this section (more specifically, corollary 3.19 and lemma 3.20, applied in $V[G_\alpha]$), for every $0 \leq j < m$,

$$U^j_\alpha = \left[\xi \mapsto U^j_\xi\right]_{U_\alpha}$$

(where U^j_ξ exists for a set of ξ-s in U_α, since $j < m$, and the function which maps each $\xi < \alpha$ to U^j_ξ is definable in V by the induction hypothesis). Thus the sequence $\langle U^0_\alpha, \ldots, U^{m_\alpha - 1}_\alpha, U^{m_\alpha}_\alpha = U_\alpha \rangle$ is definable over V from $\langle\langle U^0_\xi, \ldots, U^{m_\xi - 1}_\xi, U^{m_\xi}_\xi \rangle : \xi \in \Delta \cap \alpha \rangle$ and U_α. $\square$

Now let us proceed to the proof of the theorem. We prove by induction on $\alpha \leq \kappa^*$ that $j_\alpha : V \to M_\alpha$ is definable in V. Fix $\alpha < \kappa^*$ and assume that $j_\alpha : V \to M_\alpha$ has been defined in V. Let us define the measure $\mathcal{E}_\alpha$.

We use below the usual notations: for some $\alpha_1 < \ldots < \alpha_k < \alpha$ and $h \in V$,

$$\mu_\alpha = j_\alpha(h)\left(j_{1,\alpha}\left([Id]^{'}_0\right), j_{\alpha_1+1,\alpha}([Id]_1), \ldots, j_{\alpha_k+1,\alpha}([Id]_k)\right)$$

(for sake of definability, we can use the least $\langle \alpha_0, \ldots, \alpha_k \rangle$ and h, taken with respect to a prescribed well order of V_λ for λ high enough). For every $\xi \in \Delta$, let $\mathcal{E}(\xi)$ be the measure on $[\xi]^{m_\xi - 1}$ which corresponds to the sequence–

$$U^0_\xi \lhd U^1_\xi \lhd \ldots \lhd U^{m_\xi - 1}_\xi$$

Since $\vec{\mathcal{U}}$ belongs to V, the mapping $\xi \mapsto \mathcal{E}(\xi)$ belongs to V as well, by the claim. By corollary 3.19, for every $\alpha < \kappa^*$,

$$\mathcal{E}_\alpha = j_\alpha\left(\langle \vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k \rangle \mapsto \mathcal{E}(h\left(\vec{\xi}, \vec{\nu}_1, \ldots, \vec{\nu}_k\right))\right)\left(j_{1,\alpha}\left([Id]^{'}_0\right), j_{\alpha_1+1,\alpha}([Id]_1), \ldots, j_{\alpha_k+1,\alpha}([Id]_k)\right)$$

and this definition can be carried inside V. $\square$

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