Two grounds for the same Prikry sequence

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Abstract

We show that a Prikry sequence can have several grounds, answering a question of Goldberg and Ben-Neria. This is part of a broader range of applications of products and iterations of forcings with nonstationary support.

1 Introduction

For a normal measure U on a measurable cardinal κ, denote by $\mathbb{P}_U$ the standard Prikry forcing with respect to U. Recall that whenever $G \subseteq \mathbb{P}_U$ is generic over a ground model V, G induces an associated generic Prikry sequence,

$$\bigcup\{t \in [\kappa]^{<\omega} : \exists A \in U \ \langle t, A\rangle \in G\}.$$

The Prikry sequence associated to a $\mathbb{P}_U$-generic set is an ω-sequence that is almost contained in every set in U.

Conversely, working in an arbitrary ZFC model V, an ω-sequence $\langle \kappa_i : i < \omega\rangle \in V$ is a Prikry sequence over some inner model $V' \subseteq V$ if $\kappa := \sup_{i<\omega} \kappa_i$ is a measurable cardinal in V', $U \in V'$ is a normal measure on κ, and $\{\kappa_i : i < \omega\}$ is almost contained in every set in U. In this case, by the Mathias criterion (see [1, Theorem 1.12]), $\langle \kappa_i : i < \omega\rangle$ is the generic Prikry sequence associated with a $\mathbb{P}_U$-generic filter over V'.

Ben-Neria and Goldberg asked whether a single Prikry sequence can be generic over two distinct grounds. In this note, we give a positive answer to this question.

Theorem 1.1. *Assume GCH and the existence of a measurable cardinal. Then there exists a ZFC model V^* and an increasing sequence $\langle \kappa_i : i < \omega\rangle \in V^*$, such that V^* has two distinct inner models V_0 and V_1 satisfying the following properties:*

- $\kappa = \sup_{i<\omega} \kappa_i$ *is measurable in both V_0 and V_1.*
- *There exists a normal measure $\mathcal{W}_0 \in V_0$ such that $\langle \kappa_i : i < \omega\rangle$ is $\mathbb{P}_{\mathcal{W}_0}$-generic over V_0.*

- *There exists a normal measure $\mathcal{W}_1 \in V_1$ such that $\langle \kappa_i \colon i < \omega \rangle$ is $\mathbb{P}_{\mathcal{W}_1}$-generic over V_1.*
- $V^* = V_0[\langle \kappa_i \colon i < \omega \rangle] = V_1[\langle \kappa_i \colon i < \omega \rangle]$.

2 Preliminaries

We present our main forcing and a series of Lemmas before the proof of Theorem 1.1.

Definition 2.1. *Denote $I = \{\alpha < \kappa \colon \alpha \text{ is inaccessible}\}$. We say that a set $A \subseteq I$ is nowhere stationary if for every regular cardinal α, $A \cap \alpha$ is a nonstationary subset of α. Let κ be a Mahlo cardinal. For every $\alpha \leq \kappa$, let*

$$\mathbb{P}_\alpha = \{f \colon \alpha \to 2 \colon f \text{ is a partial function}, \ \mathrm{dom}(f) \subseteq I \cap \alpha \text{ is nowhere stationary}\}.$$

Throughout this note we let $\mathbb{P} = \mathbb{P}_\kappa$. An equivalent presentation of $\mathbb{P}$ is as a nonstationary support product forcing, $\mathbb{P} = \prod^{NS}_{\alpha \in I} \mathbb{Q}_\alpha$, where, for each $\alpha < \kappa$, $\mathbb{Q}_\alpha = \{0_{\mathbb{Q}_\alpha}, 0, 1\}$ is an atomic forcing (in which "0", "1" are incompatible elements). We refer to [2] for the definition of nonstationary support product forcing ([2, Section 1]), the proof that $\mathbb{P}$ preserves cardinals ([2, Corollary 1.6]), and the proof of the following Fusion Lemma ([2, Lemma 1.3]).

Lemma 2.2. *(Fusion Lemma) Let $p \in \mathbb{P}$ be a condition, and $\langle d(\alpha) \colon \alpha < \kappa \rangle$ a sequence of dense open subsets of $\mathbb{P}$. Then there exists $p^* \geq^* p$ and a club $C \subseteq \kappa$ such that, for every $\alpha \in C$,*

$$\{r \in \mathbb{P}_{\alpha+1} \colon r \cup (p^* \setminus \alpha + 1) \in d(\alpha)\}$$

is a dense subset of $\mathbb{P}_{\alpha+1}$ above $p^* \restriction (\alpha + 1)$.

Lemma 2.3. *Let κ be a measurable cardinal, and fix a normal measure $U \in V$ on κ. Let $G \subseteq \mathbb{P}$ be generic over V. In $V[G]$, κ remains measurable. Furthermore, for each $k < 2$, the set*

$$U \cup \{\alpha < \kappa \colon G(\alpha) = k\}$$

generates a normal ultrafilter on κ in $V[G]$.

Proof. (Sketch; the complete argument can be found in [2, Lemma 2.3]) For every $k \in \{0, 1\}$,

$$H_k = \{q \in j_U(\mathbb{P}) \colon \exists p \in G \, (q \leq j_U(p) \cup \{\langle \kappa, k \rangle\})\}$$

is $j_U(\mathbb{P})$-generic over $M_U \simeq \mathrm{Ult}(V, U)$. In fact, any $j_U(\mathbb{P})$-generic set which contains $j_U[G]$ has the form H_k for some $k \in \{0, 1\}$. In particular, j_U lifts in at least[1] two ways to an ultrapower embedding via a normal measure; indeed, for

[1] Actually, if the forcing is being done over $V = L[U]$, or modified by adding a gap below κ (for instance, by initially forcing a Cohen subset of ω and then forcing with $\mathbb{P}$) there are exactly two normal measures on κ in the generic extension.

every $k \in \{0,1\}$, U lifts to the measure $U_k \in V[G]$ which is the normal measure derived from the lift of j_U which maps G to H_k.

We argue that each U_i is generated by U and $\{\alpha < \kappa \colon G(\alpha) = k\}$. Let $\dot{A}$ be a $\mathbb{P}_\kappa$-name for a set in U_k. By the definition of U_k, there exists $p \in G$ such that $j_U(p) \cup \{\langle \kappa, k\rangle\} \Vdash \check{\kappa} \in j_U(\dot{A})$. Thus,

$$\{\alpha < \kappa \colon p \cup \{\langle \alpha, k\rangle\} \Vdash \check{\alpha} \in \dot{A}\} \cap \{\alpha < \kappa \colon G(\alpha) = k\} \subseteq A$$

and the set $\{\alpha < \kappa \colon p \cup \{\langle \alpha, k\rangle\} \Vdash \check{\alpha} \in \dot{A}\}$ belongs to U. $\square$

3 Proof of the main theorem

Proof of Theorem 1.1. In V, let U be a normal measure on κ. Let $\mathbb{P}$ be the forcing from definition 2.1. Let $G \subseteq \mathbb{P}$ be generic over V, and denote $V_0 = V[G]$. Let $\mathcal{W}_0 \in V_0 = V[G]$ be the measure generated by $U \cup \{\{\alpha < \kappa \colon G(\alpha) = 0\}\}$ as in Lemma 2.3.

Force with $\mathbb{P}_{\mathcal{W}_0}$ over $V[G]$, and let $\langle \kappa_n \colon n < \omega\rangle$ be a generic Prikry sequence for $\mathcal{W}_0$ over $V[G]$. By removing an initial segment from the sequence, we can assume that for every $n < \omega$, $G(\kappa_n) = 0$ (since $\{\kappa_n \colon n < \omega\}$ is almost contained in $\{\alpha < \kappa \colon G(\alpha) = 0\} \in \mathcal{W}_0$).

Our goal now is to find, in $V_0[\langle \kappa_i \colon i < \omega\rangle]$, a $\mathbb{P}$-generic set G' over V, which induces an inner model $V_1 = V[G']$ of $V_0[\langle \kappa_i \colon i < \omega\rangle]$.

Work in $V_0[\langle \kappa_i \colon i < \omega\rangle]$. For every $p \in \mathbb{P}_\kappa$, let p' be the function obtained from p by switching the bit $p(\kappa_n)$, for every $n < \omega$ for which $\kappa_n \in \mathrm{dom}(p)$. A key remark is that only finitely many bits in $\mathrm{dom}(p)$ are changed, because $\langle \kappa_n \colon n < \omega\rangle$ is almost contained in a club disjoint from $\mathrm{dom}(p)$. Therefore, for every $p \in V$, $p' \in V$ as well.

In $V_0[\langle \kappa_n \colon n < \omega\rangle]$ define

$$G' = \{p' \colon p \in G\}.$$

Lemma 3.1. *$G' \subseteq \mathbb{P}_\kappa$ is generic over V.*

We will need some notations and a technical argument that involves fusion for the proof. Fix $\alpha \leq \kappa$, $n < \omega$ and sequences $\vec{\beta} = \langle \beta_0, \ldots, \beta_{n-1}\rangle$, $\vec{k} = \langle k_0, \ldots, k_{n-1}\rangle$ such that $\beta_0 < \ldots < \beta_{n-1} < \alpha$ and for each $i < n$, $k_i \in \{0,1\}$. Let $r \in \mathbb{P}_\alpha$ be some condition. Let $r^{\vec{\beta},\vec{k}} \in \mathbb{P}_\alpha$ be the condition obtained from r by adding $\beta_0, \ldots, \beta_{n-1}$ to the support (if needed) and setting $r^{\vec{\beta},\vec{k}}(\beta_i) = k_i$ for every $i \leq n$, and $r^{\vec{\beta},\vec{k}}(\xi) = r(\xi)$ for every $\xi \in \mathrm{dom}(r) \setminus \{\beta_0, \ldots, \beta_{n-1}\}$.[2]

In our proof of Lemma 3.1, the main property that we will have to verify is that for every dense open $D \subseteq \mathbb{P}_\kappa$, $D \cap G' \neq \emptyset$. For that, we will use the following technical lemma.

[2]We will actually be interested in the case where $\vec{k}$ is a sequence of 1's, but we chose to work in this more general context.

Lemma 3.2. *Let $D \subseteq \mathbb{P}_\kappa$ be a dense open subset. Then D^* is dense, where D^* is the set of conditions $q \in \mathbb{P}_\kappa$ for which there exists a club $C \subseteq \kappa$ such that for every $\alpha \in C$ and $\vec{\beta} = \langle \beta_0, \ldots, \beta_{n-1} \rangle \in [\alpha]^{<\omega}$,*

$$e(\alpha, \vec{\beta}, q) = \{r \in \mathbb{P}_{\alpha+1} \colon \forall \vec{k} = \langle k_0, \ldots, k_{n-1} \rangle \in \{0,1\}^n,\ r^{\vec{\beta},\vec{k}} \cup (q \setminus (\alpha+1)) \in D\}$$

is a dense open subset of $\mathbb{P}_{\alpha+1}$ above $q \restriction (\alpha+1)$.

Proof. For every $\alpha < \kappa$, let

$$\begin{aligned} d(\alpha) = \{p \in \mathbb{P} \colon \text{ for every } \vec{\beta} \in [\alpha]^{<\omega}, \text{ the set} \\ \{r \in \mathbb{P}_{\alpha+1} \colon \forall \vec{k} \in \{0,1\}^{\mathrm{lh}(\vec{\beta})} \left(r^{\vec{\beta},\vec{k}} \cup (p \setminus (\alpha+1)) \in D\right)\} \\ \text{is a dense open subset of } \mathbb{P}_{\alpha+1} \text{ above } p \restriction (\alpha+1)\}. \end{aligned}$$

First, let us explain why it suffices to prove that $d(\alpha) \subseteq \mathbb{P}$ is dense open for every $\alpha < \kappa$. Assume that this has been proved. Fix a condition $p \in \mathbb{P}$. We argue that there exists $q \geq p$ in D^*. Indeed, by the Fusion Lemma 2.2, there exists $q \geq p$ and a club $C \subseteq \kappa$ such that, for every $\alpha \in C$,

$$\{r \in \mathbb{P}_{\alpha+1} \colon r \cup (q \setminus (\alpha+1)) \in d(\alpha)\}$$

is dense above $q \restriction (\alpha+1)$. Fix $\alpha \in C$ and $\vec{\beta} \in [\alpha]^{<\omega}$. We argue that the set $e(\alpha, \vec{\beta}, q)$ is a dense open subset of $\mathbb{P}_{\alpha+1}$ above $q \restriction (\alpha+1)$. Since D is open, $e(\alpha, \vec{\beta}, q)$ is open, so we concentrate on density. Assume that $r \geq q \restriction (\alpha+1)$. Since $\alpha \in C$, we can find $r' \geq r$ such that $r' \cup (q \setminus (\alpha+1)) \in d(\alpha)$. By the definition of $d(\alpha)$, we can find $r'' \geq r'$ (which depends on $\vec{\beta}$) such that for every $\vec{k} \in \{0,1\}^{\mathrm{lh}(\vec{\beta})}$,

$$r''^{\vec{\beta},\vec{k}} \cup (q \setminus (\alpha+1)) \in D.$$

It follows that $r'' \in e(\alpha, \vec{\beta}, q)$ and extends $q \restriction (\alpha+1)$, as desired.

Thus, it remains to prove that for every $\alpha < \kappa$, $d(\alpha)$ is dense open. Fix $\alpha < \kappa$. As before, it's clear that $d(\alpha)$ is open, so we concentrate on density. Fix $p \in \mathbb{P}$ and an enumeration $\langle \vec{\beta}_i \colon i < \alpha \rangle$ of $[\alpha]^{<\omega}$. We construct an increasing sequence of conditions $\langle s_i \colon i < \alpha \rangle \subseteq \mathbb{P} \setminus (\alpha+1)$ extending $p \restriction (\alpha+1)$, such that for every $i < \alpha$, the set

$$\{r \in \mathbb{P}_{\alpha+1} \colon \forall \vec{k} \in \{0,1\}^{\mathrm{lh}(\vec{\beta}_i)} \left(r^{\vec{\beta}_i,\vec{k}} \cup s_i \in D\right)\}$$

is a dense open subset of $\mathbb{P}_{\alpha+1}$ above $p \restriction (\alpha+1)$. Once the sequence $\langle s_i \colon i < \alpha \rangle$ is constructed, take $s^* = \bigcup_{i<\alpha} s_i$ (note that $\mathbb{P} \setminus (\alpha+1)$ is sufficiently closed to ensure that $s^* \in \mathbb{P} \setminus (\alpha+1)$). Then $q = (p \restriction (\alpha+1)) \cup s^*$ is an extension of p in $d(\alpha)$, as desired.

For the construction of $\vec{s} = \langle s_i \colon i < \alpha \rangle$, assume that $i < \alpha$ and $\vec{s} \restriction i$ was constructed. Our goal is to construct a condition s_i with the above-mentioned property. First, let $s_i^* = \bigcup_{j<i} s_i$ (and, in the case where $i = 0$, let $s_0^* =$

$p \setminus (\alpha+1)$). Enumerate $\langle r_n \colon n < |\alpha|^+\rangle$ all the conditions in $\mathbb{P}_{\alpha+1}$ which extend $p \restriction (\alpha+1)$. We define an increasing sequence of conditions $\langle s_i^n \colon n < |\alpha|^+\rangle \subseteq \mathbb{P} \setminus (\alpha+1)$ extending s_i^* such that, for every $n < |\alpha|^+$, there exists an extension $r'_n \geq r_n$ in $\mathbb{P} \restriction (\alpha+1)$ such that for every $\vec{k} \in \{0,1\}^{\mathrm{lh}(\beta)_i}$,

$$(r'_n)^{\vec{\beta}_i, \vec{k}} \cup (s_i^n) \in D.$$

Once the sequence $\langle s_i^n \colon n < |\alpha|^+\rangle$ has been constructed, let $s_i = \bigcup_{n<|\alpha|^+} s_i^n$, and note that $s_i \in \mathbb{P} \setminus (\alpha+1)$ is as desired. Thus, it remains to define the sequence $\langle s_i^n \colon n < |\alpha|^+\rangle$. Assume that $\langle s_i^m \colon m < n\rangle$ has been constructed for some $n < |\alpha|^+$. Let $s_i^{*n} = \bigcup_{m<n} s_i^m \in \mathbb{P} \setminus (\alpha+1)$. We argue that there exists $r' \in \mathbb{P}_{\alpha+1}$ and $s' \in \mathbb{P} \setminus (\alpha+1)$ such that $r' \geq r_n$, $s' \geq s_i^{*n}$, and for every $\vec{k} \in \{0,1\}^{\mathrm{lh}(\vec{\beta}_i)}$,

$$(r')^{\vec{\beta}_i, \vec{k}} \cup s' \in D.$$

Once we prove that such r', s' exist, we take $r'_n = r'$ and $s_i^n = s'$.

Thus, it remains to construct r', s' as above. Fix an enumeration $\langle \vec{k}_\ell \colon \ell < \ell^*\rangle$ of $\{0,1\}^{\mathrm{lh}(\vec{\beta}_i)}$ (here, $\ell^* = 2^{\mathrm{lh}(\vec{\beta}_i)}$, and recall that $\vec{\beta}_i$ is finite and ℓ^* is a natural number). We construct a pair of finite sequences of conditions, $\vec{R} = \langle R_\ell \colon \ell < \ell^*+1\rangle \subseteq \mathbb{P}_{\alpha+1}$ and $\vec{S} = \langle S_\ell \colon \ell < \ell^*+1\rangle \subseteq \mathbb{P} \setminus (\alpha+1)$ such that $R_0 = r_i$, $S_0 = s_i^{*n}$, the sequence $\vec{S}$ is increasing, and the sequence $\vec{R}$ satisfies that each $R_{\ell+1}$ extends $(R_\ell)^{\vec{\beta}_i, \vec{k}_\ell}$. For the construction of the sequences, assuming that $\ell < \ell^*$ and R_ℓ, S_ℓ are given, let $R_{\ell+1} \in \mathbb{P}_{\alpha+1}, S_{\ell+1} \in \mathbb{P} \setminus (\alpha+1)$ be such that $(R_\ell)^{\vec{\beta}_i, \vec{k}_\ell} \cup S_\ell \leq R_{\ell+1} \cup S_{\ell+1} \in D$. Finally, let $s' = \bigcup_{\ell<\ell^*+1} S_\ell$, and let r' be the condition such that:

- $\mathrm{dom}(r') = \bigcup_{\ell<\ell^*+1} \mathrm{dom}(R_\ell) = \mathrm{dom}(R_{\ell^*})$.
- for every β in $\vec{\beta}_i$, $r'(\beta) = r_i(\beta)$ (and, if $\beta \notin \mathrm{dom}(r_i)$, just take $r'(\beta) = 0$).
- for every other value of $\beta \in \mathrm{dom}(R_{\ell^*})$, take $r'(\beta) = R_{\ell^*}(\beta)$.

Note that $r' \geq r_i$ and s' are as desired, since for every $\ell < \ell^*$,

$$(r')^{\vec{\beta}_i, \vec{k}_\ell} \cup s' \geq R_{\ell+1} \cup S_{\ell+1} \in D.$$

This concludes the construction of the sequence $\langle s_i^n \colon n < |\alpha|^+\rangle$, which concludes the proof. □

We are now ready to prove Lemma 3.1.

Proof of Lemma 3.1. Let $D \subseteq \mathbb{P}_\kappa$ be dense open. We argue that $G' \cap D \neq \emptyset$.

Let D^* be as in Lemma 3.2. Since D^* is dense, we may pick a condition $q \in G \cap D^*$. Let C be a club which is both disjoint from $\mathrm{dom}(q)$ and witnesses that $q \in D^*$.

Let C^* be the club of limit points of C. Since $C^* \in U$, there exists $n_0 < \omega$ such that for every $n \geq n_0$, $\kappa_n \in C^*$. In particular, $\kappa_n \in C$ and thus $\kappa_n \notin$

dom(q) for all $n \geq n_0$. Let $\alpha = \min(C \setminus \kappa_{n_0-1}+1)$. Since κ_{n_0} is a limit point of C, $\alpha < \kappa_{n_0}$. Denote $\vec{\beta} = \langle \kappa_0, \ldots, \kappa_{n_0-1}\rangle \in [\alpha]^{<\omega}$. By the definition of D^*, the set $e(\alpha, \vec{\beta}, q)$ is dense open in $\mathbb{P}_{\alpha+1}$. Thus, there exists $r \in G \restriction (\alpha+1)$ such that, for every $\vec{k} \in \{0,1\}^{n_0}$,

$$r^{\vec{\beta},\vec{k}} \cup (q \setminus (\alpha+1)) \in D.$$

By increasing r in $G \restriction (\alpha+1)$ we can assume that $\{\kappa_0, \ldots, \kappa_{n-1}\} \subseteq \mathrm{dom}(r)$.

Let $\vec{k}$ be a sequence of 1's. Recall that $\langle \kappa_i \colon i < \omega\rangle$ was chosen such that $G(\kappa_i) = 0$ for every $i < \omega$. In particular, the condition $r^{\vec{\beta},\vec{k}} \cup (q \setminus (\alpha+1))$ is obtained from $r \cup (q \setminus (\alpha+1)) \in G$ by switching the bits at coordinates $\langle \kappa_0, \ldots, \kappa_{n_0-1}\rangle$ from 0 to 1. Since $\alpha < \kappa_{n_0}$, the Prikry points κ_n for $n \geq n_0$ are outside the support of $r \cup (q \setminus (\alpha+1))$. Overall, it follows from the definition of G' that

$$r^{\vec{\beta},\vec{k}} \cup (q \setminus (\alpha+1)) \in G'.$$

Therefore $G' \cap D \neq \emptyset$, as desired. □

Denote $V_1 = V[G']$. Let $\mathcal{W}_1$ be the normal measure generated by $U \cup \{\alpha < \kappa \colon G'(\alpha) = 1\}$ in $V[G']$ (see Lemma 2.3 for the proof that a normal measure is generated this way in $V[G']$).

We argue that $\langle \kappa_n \colon n < \omega\rangle$ is generic for Prikry forcing with $\mathcal{W}_1$ over $V[G']$. By the Mathias criterion, it suffices to prove that $\langle \kappa_n \colon n < \omega\rangle$ is almost contained in $\{\alpha < \kappa \colon G'(\alpha) = 1\}$. This is indeed true: recall that for every $n < \omega$, $G(\kappa_n) = 0$. By the definition of G', it follows that for every $n < \omega$, $G'(\kappa_n) = 1$, namely $\langle \kappa_n \colon n < \omega\rangle \subseteq \{\alpha < \kappa \colon G'(\alpha) = 1\}$, as desired.

Finally, note that $V_0 \neq V_1$, since otherwise $\langle \kappa_n \colon n < \omega\rangle$ would have belonged to $V_0 = V[G]$ (as it can be computed from G, G'). Furthermore,

$$V_0[\langle \kappa_n \colon n < \omega\rangle] = V_1[\langle \kappa_n \colon n < \omega\rangle].$$

Then $\mathcal{W}_0 \in V_0, \mathcal{W}_1 \in V_1$ and $\langle \kappa_n \colon n < \omega\rangle$ are as required in Theorem 1.1. □

We conclude this note with a remark regarding the proof technique. The main advantage of using the nonstationary support product is that the measures $\mathcal{W}_0, \mathcal{W}_1$ are generated from U and a single additional set in V_0, V_1 respectively (see Lemma 2.3). This simplifies checking whether the sequence $\langle \kappa_n \colon n < \omega\rangle$ is $\mathbb{P}_{\mathcal{W}_1}$-generic over V_1 (which was one of the final steps in the proof of Theorem 1.1). However, we are not sure that the nonstationary support is actually required for this.

Question 3.3. *Can the forcing $\mathbb{P}$ in the proof be replaced with an Easton support product?*

We conjecture that the answer is positive. Working with an Easton support would greatly simplify some aspects of the proof (for example, Lemma 3.2 would be replaced by a simpler argument), but would likely require more care in defining the measures $\mathcal{W}_0$ and $\mathcal{W}_1$.

References

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[2] Eyal Kaplan. The number of normal measures, revisited. *arXiv preprint arXiv:2507.20466*, 2025.